KISII UNIVERSITY

BSc. ACTUARIAL SCIENCE
MATHEMATICAL MODELLING

BACS 213
EXAM

DECEMBER, 2022

ATTEMPT ALL QUESTIONS IN SECTION ONE AND ANY TWO OTHER QUESTIONS IN SECTION TWO.

TIME: 2 HOURS

SECTION ONE (30 marks)

1 Question One

a) i) Consider the Newton's law of cooling which provides a mathematical model of the temperature T (t) of an object in surroundings of temperature A(t):

$$\frac{dT}{dt} = -k(T - A(t))$$

where k > 0 measures the rate that heat is absorbed (or emitted) by the object. Show that the temperature at t_2 in terms of the temperature at time t_1 can be expressed as

$$T\left(t_2\right) = A + \left[T\left(t_1\right) - A\right] e^{-k(t_2 - t_1)}$$

(3 *marks*)

ii) A body is found in a cold room (temperature 5 ^0C) at 3 *p.m.* and its temperature then is 19 ^0C. An hour later its temperature has dropped to 15

^0C Use Newton's law of cooling to estimate the time of death, assuming that body temperature is 37 ^0C. (5 *marks*)

b) Claims arrive at an insurance company according to a Poisson process with rate per week. Assume time is expressed in weeks.

i) Show that, given that there is exactly one claim in the time interval $[t, t+s]$, the time of the claim arrival is uniformly distributed on $[t, t+s]$.

ii) State the joint density of the holding times $T_0, T_1, ..., T_n$ between successive claims.

iii) Show that, given that there are n claims in the time interval $[0, t]$, the number of claims in the interval $[0, s]$ for $s < t$ is binomial with parameters n and s/t. (7 *marks*)

c) Find the probability generating function of a random variable, X, assumed to follow a poisson distribution with parameter β. Hence give expression for

i) $Prob(X = 3)$

ii) $Prob(X = 5)$ (7 *marks*)

d) The share price, in pence, of a certain company is monitored over an $8 - year$ period. The result are shown in the table below:

Time (years)	0	1	2	3	4	5	6	7	8
Price	100	130	182	250	320	450	600	820	1,100

An actuary fits the following simple linear regression model to the data:

$$y_i = \alpha + \beta x_i + e_i \quad i = 0, 1, 2, \ldots, 8.$$

where $\{e_i\}$ are independent normal random variables with mean zero and variance σ^2.

(i) Determine the fitted regression line in which the price is modelled as the response variable and the time as the explanatory variable (2 *marks*)

(ii) Obtain the 99% confidence interval for β, the true underlying slope parameter. (3 *marks*)

(iii) State the SS_{TOT} and calculate its partition into SS_{REG} and SS_{RES} (3 *marks*)

SECTION TWO (40 marks)

2 Question Two

a) The population of Great Britain and Ireland in 1801, 1851 and 1901 can be found in the results of the Census for each of those years:

Year	*Population*
1801	16 345 646
1851	27 533 755
1901	41 609 091

Estimate the expected population size for years 2001 using

i) Malthus' population model

ii) Logistic model

The true figure for the population in 2001 was 63 million. Comment on the values obtained above. (10 *marks*)

b) An Actuary wishes to investigate the significance of mixed poisson model. He considers a conditional poisson distribution with parameter β. He further models the variability of the parameter using a *Gamma I* distribution with parameters α *and* λ.

i) Construct the mixed poisson model and deduce the mean and variance.

Consider the data below:

X	0	1	2	3	4	5	6	7	8	9
Observed	21	41	32	16	19	8	4	1	2	1

ii) Compute the expected frequencies for the poisson and Negative Binomial distributions. Using the Chi Square test and the loglikelihood, comment on the two models. (10 *marks*)

3 Question Three

a) i) Prove that, under Gompertz's Law, the probability of survival from age x to age $x+t$, ${}_tp_x$, is given by:

$${}_tp_x = \left[\exp\left(\frac{-B}{\ln c}\right)\right]^{c^x(c^t-1)}$$

(4 *marks*)

For a certain population, estimates of survival probabilities are available as follows:

$$
\begin{aligned}
{}_1p_{50} &= 0.995 \\
{}_1p_{50} &= 0.989
\end{aligned}
$$

ii) Calculate values of B and c consistent with these observations. (3 *marks*)

iii) Comment on the calculation performed in (ii) compared with the usual process for estimating the parameters from a set of crude mortality rates. (3 *marks*)

b) A mathmatician believes the conditional distribution, $f(x|\beta)$, is an exponential distribution with parameter β. She further models the variability of the scale parameter β with a gamma ditribution, i.e, $\beta \sim Gamma(\alpha, \lambda)$. Find:

(i) the mixed model and identify the distribution. (4 *marks*)

(ii) the mean and variance of the mixed model. (6 *marks*)

4 Question Four

a) A life insurance company wishes to value some benefit payable at the end of year of death to a life aged exactly 60 years. The contract period is five years and the benefit paid in year k is expressed as

$$B_k = 100000(1 - 0.25(k-1));\ k = 1, 2, 3, 4, 5$$

The office assumes the following mortality

$age,\ x$	$No.\ of\ persons,\ l_x$	$deaths,\ d_x$
60	90085	656
61	89429	736
62	88693	825
63	87868	923
64	86945	1029
65	859I6	1144

Compute the Expected present value of the benefits. (9 *marks*)

b) A Statistical analyst believes the poisson distribution with parameter

λ is a suitable fit for claims arising from a certain insurance policy. From the policy, the following claims distribution is available in the insurer's file

Number of Claims	*Frequency*
2	180
3	60
4	20
5	10
≥ 6	0

Only data concerning those policies on which claims were made can be used in estimation of claim rate λ.

Using

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!\left(1 - e^{-\lambda} - \lambda e^{-\lambda}\right)} \quad x = 2, 3, 4, \ldots$$

i) show that both the method of moments estimate and the MLE of λ satisfy the equation $\lambda\left(1 - e^{-\lambda}\right) = \bar{x}\left(1 - e^{-\lambda} - \lambda e^{-\lambda}\right)$, where $\bar{x}$ is the mean number for claims for policies that have at least two number of claims. (7 *marks*)

(*ii*) Solve this equation by any means for the given data and calculate the resulting estimate for λ to two decimal places.(3 *marks*)

(*iii*) Hence, estimate the percentage of all policies with at most one claim during the year.(1 *mark*)

5 Question five

a) A car magazine published an article exploring the relationship between the mileage (in units of 1,000 miles) and the selling price (in units of £1,000) of used cars. The following data were collected on 10 four year old cars of the same make.

Car	1	2	3	4	5	6	7	8	9	10
Mileage, x	42	29	51	46	38	59	18	32	22	39
Price, y	5.3	6.1	4.7	4.5	5.5	5.0	6.9	5.7	5.8	5.9

i) Determine the correlation coefficient between x and y. Comment on its value.

A linear model of the form $y = \alpha + \beta x + \varepsilon$ is fitted to the data , where the error terms (ε) independently follow a $N(0, \sigma^2)$ distribution, with σ^2 being an unknown parameter.

ii) Determine the fitted line of the regression model.

iii) Determine a 95% confidence interval for β.

iv) Calculate the estimated difference in the selling prices for cars that differ in mileage by 5,000 miles. (15 *marks*)

b) You have decided to take a *one million* loan which pays level annual payments at an interest rate of 12% *pa effective*. Calculate the level premium for the loan. (5 *marks*) .