MATH 312: REAL ANALYSIS 1

STREAM: Y3 S1 **TIME:2 HOURS**

INSTRUCTIONS

1.Do not write anything on this question paper.

2.Answer question ONE (compulsory) and any other TWO questions.

QUESTION ONE (30 MARKS)

(a) Prove by mathematical induction that

$\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for all natural numbers $n \geq 2$. (5mks)

(b) Prove that $\sqrt{5}$ is irrational . (5mks)

(c) Prove that if $0 < a < b$ then $a^2 < b^2$ for all $, b \in \mathbb{R}$. (6mks)

(d) (i) If m is an odd integer, show that its square is an odd integer. (3mks)

(ii) Let a and b be rational numbers. Prove that between a and b there is another rational number. (3mks)

(e) Define the following :

(i) Inductive set (2mks)

(ii) Bounded set (2mks)

(f) Let x and y be real numbers such that $x \leq y + \varepsilon$ for every $\varepsilon > 0$. Then prove that $x \leq y$. (5mks)

QUESTION TWO (20 MARKS)

(a) Distinguish between the following terms as used for a set of real numbers:

(i) Infimum and minimum element (2mks)

(ii) Supremum and maximum element (2mks)

(b) Prove that every subset of the set of natural numbers which is non-empty has a minimum element. (5mks)

(c) Show that the set $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is bounded below by infimum 0 and has no minimum element. (5mks)

(d) Let T be a non-empty set of real numbers with supremum b . Prove that for every element $a < b$ there exists an element $t \in T$ such that $a < t < b$ (6mks)

QUESTION THREE (20 MARKS)

(a) (i)Show that $|a| \leq b$ if and only if $-b \leq a \leq b$. (5mks)

(ii) Prove that $|x + y| \leq |x| + |y|$ where x and y are real numbers. (5mks)

(b) (i) If x_n and y_n are two converging sequences as $n \to \infty$ with limits l and h respectively, prove that $x_n + y_n \to l + h$. (5mks)

(ii) Show that if $\{x_n\}$ is a monotonically increasing sequence of real numbers and is bounded above, then x_n converges to its spremum. (5mks)

QUESTION FOUR (20 MARKS)

(a) (i)State the intermediate value theorem . (2mks)

(ii) Use the intermediate value theorem to prove that $f(x) = 2x^2 - 5$ has a

solution on the interval $[-1,3]$. (3mks)

(b) Prove that $\lim_{x \to 2} 4x + 2 = 10$ using $\varepsilon - \delta$ definition. (4mks)

(c) (i)Define a metric space (3mks)
(ii) Let $x, y \in \mathbb{R}$ and $d(x, y) = |x - y|$. Show that $(\mathbb{R}, d)$ is a metric space. (6mks).

(iii) Find $N(-5,3)$ in $(\mathbb{R}, d)$. (2mks)

QUESTION FIVE (20 MARKS)

(a) (i) Prove that if f is defined on (a, b) except possibly at $c \in (a, b)$ then

$f(x) \to l$ as $x \to c$ if and only if $f(x) \to l$ as $x \to c^-$ and $f(x) \to l$

as $\to c^+$. (4mks).

(b) Let f, g and h be functions defined on (a, b) except possibly at $x = c$. If

$g(x) \to L$ as $x \to c$ and $h(x) \to L$ as $x \to c$ and $g(x) \leq f(x) \leq h(x)$, show that

$f(x) \to L$ as $x \to c$. (5mks)

(c) (i)Prove that $f(x) = x^{-1}$ diverges to $+\infty$ as $x \to 0^+$ (3mks)

(ii) Show that if $f_n(x) = x^n$ for $0 < x < 1$, then $f_n(x) \to f(x)$ when $f(x) = 0$.

(4mks)

(d) Using the definition of continuity at a point, prove that $f(x) = 3x - 5$ is

continuous at $x = 3$. (4mks)