KISII UNIVERSITY

EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

MATH 821- OPERATOR THEORY I

STREAM- MSC Y2SI

TIME: 3 HOURS

INSTRUCTIONS: ANSWER QUESTION ONE COMPULSORY AND ANY OTHER TWO QUESTIONS

QUESTION ONE (30 MARKS)

a) i) Let $A = \begin{pmatrix} 1 & 3 \\ -5 & 9 \end{pmatrix}$. Determine $\delta(A)$, eigen space of A and $\rho(A)$ (9mks)

ii) What is a regular value of an operator T? (1 mk)

b) Suppose $T \in B(X)$ where $X \neq \{0\}$ is a Hilbert space. Prove that

i) Ker T=$(ImT^{*})^{\perp}$ (4mks)

ii) T is invertible iff T*is invertible (4mks)

c) Prove that the spectral radius and the norm of a self adjoint operator T on X coincide (6mks)

d) i) What is a linear projection operator? (1mk)

ii) Prove that for any projection P on a Hilbert space H, $\|p\| \leq 1$,

$\|p\| = 1$ if P(H)$\neq \{0\}$ (5mks)

<u>QUESTION TWO (20 MARKS)</u>

a) i) If S is a bounded linear operator on a Banach space X and $\|S\| < |\lambda|$,

$S_\lambda = (\lambda I - S)^{-1}$ is a bounded operator. Prove that $S_\lambda = \sum_{n=0}^{\infty} \frac{S^n}{\lambda^{n+1}}$ (10 mks)

ii) State the Lax-Milgram lemma (2mks)

b) Define the spectrum of an operator T hence find the spectrum of the matrix

M= $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for $b \neq 0$ (6mks)

<u>QUESTION THREE (20 MARKS)</u>

a) Let T$\in B(X, X)$ where X is a Banach space. Prove that if $\|T\| < 1$

then $(I - T)^{-1}$ exists as a bounded linear operator on X and

$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j = I + T + T^2 + ...$ (10mks)

b) Let T be a normal operator, prove that $T_x = \lambda_x$ iff $T^*x = \overline{\lambda}x$ (10mks)

QUESTION FOUR (20 MARKS)

a) Prove that the spectrum of a bounded self adjoint linear operator T:H→H on a complex hilbert space H is real (5mks)

b) Let H be a Hilbert space, prove that $p = p_1 p_2$ is a projection on H iff the $p_1 p_2 = p_2 p_1$ (5mks)

b) Find a linear operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which is indempotent but not self adjoint (3mks)

c) State and proof the Hellinger-Toeplitz theorem (7mks)