ORTHOGONALITY OF FINITE OPERATORS IN NORMED SPACES

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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Dedication

To my loving parents, Mr. Peter Orina and Mrs. Susan Orina.

Abstract

Orthogonality of operators in Hilbert spaces is a notion that has been studied for a duration of some time by many mathematicians such as Oleche, Okelo, Agure and many others. Many researchers have obtained great results concerning orthogonality of operators in normed spaces especially of elementary operators but this has not been fully investigated particulary orthogonality of finite operators in normed spaces. In this study, we considered finite operators and characterized their orthogonality. The objectives of this study are to: Characterize finiteness of elementary operators, establish orthogonality conditions for finite elementary operators and determine Birkhoff-James orthogonality for finite elementary operators. The methodology involved the use of Gram Schmidt procedure, Berberian Technique, Putnam Fuglede property, use of known inequalities such as Triangle inequality, Minkowski's inequality, Hölder's inequality, Cauchy Schwarz inequality and Bessel's inequality. We also used technical approaches such as Tensor product and Direct sum decomposition. Concerning finite elementary operators we showed that the elementary operators(Jordan elementary operator, generalized derivation, inner derivation, basic elementary operator) are finite. Then, regarding orthogonality conditions for finite elementary operators we proved that the range of finite elementary operators is orthogonal to its null space if the operators are contractive and finally on Birkhoff-James orthogonality for finite elementary operators we showed that the the range of finite elementary operators is orthogonal to its kernel in terms of Birkhoff-James. The results obtained are applicable in quantum theory in estimation of the distance between the identity operator and the commutators.

Contents

Index of Notations

Chapter 1

INTRODUCTION

1.1 Mathematical background

The usual definition of orthogonality of vectors of a metric space is that $q \perp h$ if and only if the inner product $\langle q, h \rangle = 0$. Orthogonality in any space that is normed can not be defined in the same way of the space endowed with an inner product because a normed space is not always an inner product space. Hence, since 1934 various concepts of orthogonality in Hilbert spaces have been studied and introduced by Birkhoff, James, Robert[3] [2] [1] [14] among others. These studies lead to several versions of orthogonality such as:

(i). Rorberts orthogonality(1934): $\|j - \gamma k\| = \|j + \gamma k\|$, for all $\gamma \in \mathbb{R}$.

(ii). Birkhoff orthogonality(1935): $\|j\| \leq \|j + \gamma k\|$, for all $\gamma \in \mathbb{R}$.

(iii). Isosceles orthogonality(1945): $\|j - k\| = \|j + k\|$.

(iv). Pythagorean orthogonality(1945): $\|j - k\|^2 = \|j\|^2 + \|k\|^2$.

(v). Singer orthogonality(1957): $j = 0 \quad or \quad k = 0 \quad or \quad \|\frac{j}{\|j\|} + \frac{k}{\|k\|}\| = \|\frac{j}{\|j\|} - \frac{k}{\|k\|}\|$.

(vi). a-isosceles orthogonality(1988): $\|j - ak\| = \|j + ak\|$.

(vii). a-pythagorean orthogonality(1988): $\|j - ak\|^2 = \|j\|^2 + a^2\|k\|^2$.

(viii). Carlsson(1961): $\sum_{k=1}^{m} a_k\|b_kx+c_ky\|^2 = 0$ where $m \geq 2$ and $a_k, b_k, c_k \in \mathbb{R}$, $\sum_{k=1}^{m} a_kb_kc_k \neq 0$, $\sum_{k=1}^{m} a_kb_k^2 = \sum_{k=1}^{m} a_kc_k^2 = 0$.

(ix). ab(1978): $\|ap + bq\|^2 + \|p + q\|^2 = \|ap + q\|^2 + \|p + bq\|^2$.

(x). a(1983): $(1 + a^2)\|q + r\|^2 = \|aq + r\|^2 + \|q + ar\|^2$.

(xi). U-isosceles(1957): either $\|q\|\|r\| = 0$, or $\|q\|^{-1}q$ is isosceles-orthogonal to $\|r\|^{-1}r$.

(xii). U-pythagorean(1986): either $\|q\|\|r\| = 0$, or $\|q\|^{-1}q$ is pythagorean-orthogonal to $\|r\|^{-1}r$.

(xiii). Area(1986): either $\|q\|\|r\| = 0$, or they are linearly independent and that $q, r, -q, -r$ cut the unit ball of their plane independently in four equivalent parts.

(xiv). Diminnie(1983): $\sup\{q(e)s(j) - q(j)s(e) : q, s \in S'\}=\|e\|\|j\|$, S' representing the unit sphere of the space of functionals of E.

A linear operator Q on a Hilbert space H is finite if $\|QX - XQ - I\| \geq 1$ for each $X \in L(H)$. William [92] showed that the algebra of finite operators involves normal operators, operators that are closed, operators with a uniformly continuous summand, and the Banach algebra with an involution satisfying the properties of adjoint originating from each and every member. The results implied the group of self-commutators is uniformly closed and that the class of operators that have a reducing subspace of finite dimension is non-uniformly dense. Elalami [39] gave a new class of finite operators using the knowledge of the reducing approximate spectrum of an operator. In this case the concept of completely finite operators was introduced. Those are operators A such that A_E is finite for any orthogonal reducing subspace E of A. For those operators Elalami [39] gave characterizations and proved that dominant operators are completely finite.

Duggal [36] improved the inequality of Du Hong-Ke to $\|QZQ - Z + J\| \geq \|Z\|$ for all operators Z. Indeed, Duggal [36] proved that the inequality of Du Hong-Ke is valid for unitary invariant norms and it was shown that the Du Hong-Ke inequality is equivalent to the Anderson inequality. In [86] Takayuki, Masatoshi and Takeaki introduced another group "class A" provided by operator inequalities that involves the group of paranormal operators and the group of log-hyponormal operators. It turned out that their results contained another proof of Ando's results in which every log-hyponormal operator is paranormal. New groups of operators similar to class A operators and paranormal operators were also introduced.

Salah [79] gave a group of finite operators of the form $S + G$ whereby $S \in L(Z)$ and G is compact. Salah [79] proved that $w_o(\delta_{S,P}) = c_o\delta(\delta_{S,P})$, where $w_o(\delta_{S,P})$, $c_o\delta(\delta_{S,P})$ denote respectively the numerical range of $\delta_{S,P}$ and the convex hull of $\delta(\delta_{S,P})$ (the spectrum of $\delta_{S,P}$) for certain operators $S, P \in L(Z)$, $\delta_{S,P}$ is the ant operator on $L(Z)$ defined by $\delta_{S,P} = SZ - ZP$ $Z \in L(Z)$. In [81] Salah characterized the operators $T \in L(H)$ and proved the range-kernel orthogonality results for the operators $Q, R \in$

$L(H)$ that are non-normal in terms of Birkoff-James. Salah [81] introduced another notion to characterize Anderson's theorem that is independent of normality through the Putnam-Fuglede property.

Bachir [15] gave results on orthogonality of dominant operators and log-hyponormal or p-hyponormal operators. Bachir [15] studied orthogonality of certain operators. The main goal was to dertermine the range-kernel orthogonality results of $\delta_{S,R}$ for some operators. Bachir [15] proved that the range of $\delta_{S,R}$ is orthogonal to the nullspace of $\delta_{S,R}$ when R^* and S is dominant is log-hyponormal or p-hyponormal. In [80] Salah proved that paranormal operators are finite and presented some examples of operators that are finite. Further study of the inequality $\|I - AX - XA\| \geq 1$ was also given. Bouzenda [83] proved that a spectraloid operator is finite and that the operator given by $A + K$ is also finite where A is convexoid and K is compact. Bouzenda [83] studied orthogonality of some operators, a new class of finite operators was given and some generalized operators were presented.

Hadia [48] presented some properties of finite operators and gave some groups of operators which are in the group of finite operators and found for which condition $A + W$ is a finite operator in $L(H \oplus H)$. Salah [78] presented another set of operators that are finite which involves the set of paranormal operators and proved that the range and null space of $\delta_{Y,Z}$ are orthogonal for a group of operators involving the group of operators that are normal. Salah and Smail [84] proved that a paranormal operator is finite and presented properties of finite operators.

Kapoor and Jagadish Prasad [59] characterized inner product spaces and provided simple results of characterizations same as the existing ones. Also, in [59] it was shown that, Isosceles orthogonality is distinctive provided the space is convex and that Pythagorean orthogonality is unique in spaces with norm structure. Bhatia and Semrl [19] showed that if Q and F are matrices such that $\|Q + zF\| \geq \|Q\|$ for all complex numbers z, then in this case Q is orthogonal to F. Bhatia and Semrl [19] found important properties for this orthogonality to hold. Some characterizations and generalizations were also obtained.

In normed spaces, Diminnie, Raymond and Edward [26] defined both pythagorean and Isosceles orthogonality and it was found that the homogeneity property holds for the orthogonalities in an inner product space. Koldobsky [56] showed that a bounded linear operator $G : Y \to Y$ is orthogonal provided that there is a product G and a positive constant. Alonso and Maria [4] studied geometric properties defined in Banach spaces of an orthogonality relation and based on the property of right angles.

Jacek [53] defined an approximate Birkhofff orthogonality relation in normed spaces. Jacek [53] compared it with that introduced by Dragomir and established few characteristics of approximate Birkhofff orthogonality. In this case, it was shown that approximate Birkhofff orthogonality in smooth space and from the semi-inner product is equivalent to approximate orthogonality. In [28] Dragoljub introduced ψ-Gateaux derivative for operators to be orthogonal to the operator in both spaces C_1 and C_∞ (nuclear and compact operators on a Hilbert spaces). Further, Dragoljub [28] applied these results to prove that there exists a normal derivation δ_A such that $\overline{ran\delta_A} \oplus ker\delta_A \neq C_1$ and a related result concerning C_∞.

Fathi [40] adopted the notion of orthogonality and established a characterization for orthogonality in the spaces $L_s^p(\mathbb{C})$, $1 \leq P < \infty$. Fathi [40] denoted $L(Q, Z)$ as the group of operators from the normed space Q to the Banach space Z. For the Hilbert spaces Q and Z it was shown that the group of completely continuous operators in $L(Q, Z)$ is the closure $L(Q, Z)$ of the algebra of operators with finite rank. That gave a more efficient description of completely continuous operators. Fathi [40] showed that this property holds whereby Q is any space that is normed and Z is a Banach space in which an orthonormal countable basis holds. Orthogonality was defined in relation to coefficient functionals. The objective was to describe a new and understandable type of orthogonality that explains in detail the structure that can be used to study different groups of functional spaces and operators.

Fathi [40] introduced another definition that includes characteristics among other things. It was shown that given (x_n) is orthonormal then (x_n) is semi orthonormal. Fathi [40] established new geometric properties for the different types. Various examples were obtained to show there is a possibility for (x_n) to be orthonormal. Fathi [40] finished by obtaining another example whereit was shown that x_i is orthogonal to x_j for all $i \neq j$ while x_n is not orthonormal. Finally, generalizations in the Banach spaces of the usual characterization of orthogonality $L_s^2(\mathbb{C})$, through inner products were obtained.

Debmalya, Kallol and Jha [34] proved that the normed space X is strictly convex such that given the elements q, r of the unit sphere S_X, $q \perp r$ means that $\|q + \lambda r\| > 1$ for every $\lambda \neq 0$. Debmalya, Kallol and Jha [34] applied this result through Birkhoff-James orthogonality to find the properties for strong orthonormality of a countable basis in a strictly convex space Z of finite dimension. Using this result Debmalya, Kallol and Jha [34] gave an estimation for lower limits of $sj + (1-s)k$, $s \in [0,1]$ and $\|k+\lambda s\|$,for complex λ, for every $j, k \in S_j$ with $j \perp_B k$. Debmalya, Kallol and Jha [34] found a condition for the existence of conjugate diameters through the points $e_1, e_2 \in S_j$ in a real 2-dimensional strict convex space. For a real strictly convex smooth space of finite dimension the concept of generalized conjugate diameters was introduced.

Hendra and Mashadi [49] discussed some concepts of orthogonalities in 2-normed spaces and their drawbacks. New definitions of orthogonality were formulated which improved the existing ones. In the standard 2-normed spaces the usual orthogonality coincide with their notions of orthogonality. Turnsek [90] characterized isometries and co-isometries in $L(Z)$ in the sense of James' orthogonality. As a result [90] Turnsek obtained a characterization of conjugate linear or surjective linear mappings $\phi : L(Z) \to L(Z)$ preserving James' orthogonality in all directions.

Madjid and Mohammad [69] introduced the notion of orthogonality constant mappings in isosceles orthogonal spaces and established stability of orthogonal constant mappings and the stability of periderized quadratic equation $q(r+s) + h(r+s) = g(r) + g(s)$ was studied. Madjid and Mohammad [69] dealt with isosceles orthogonality and in their case a normed linear space Z given that the isosceles orthogonality was referred as an isosceles orthogonal space. Shoja and Mazaheri [85] investigated some properties of the General orthogonality in Banach spaces, and obtained some results of general orthogonality in Banach spaces similar to orthogonality of Hilbert spaces. The relation between this concept in smooth spaces and sense of Birkhoff-James was also considered.

In normed spaces, Jacek [52] considered a class of linear mappings preserving this relation through Birkhoff-James orthogonality. Some related stability problems were stated. Horst [46] showed that the linear mapping from a normed space Q to a normed space R is isosceles orthogonal given that it is an isometric scalar multiple. In normed spaces, it was shown that the concept of distance that preserve maps originated from the Mazur-Uham theorem. Since Birkhoff-Orthogonality is homogenous and not symmetric whereby Isosceles orthogonality is symmetric and not homogenous, that showed that the two types of orthogonality have different properties in linear normed spaces. In inner product spaces, one could easily yield the concepts of orthogonality that yield the usual orthogonality. Precisely, the orthogonalities mean the same given an inner product space. Therefore they might have been referred as natural extention of orthogonality to normed spaces.

Horst [46] investigated that an orthogonal linear map in an inner product space is necessary an isometric scalar multiple, whereby a mapping Q preserves orthogonality provided that p is orthogonal to s means that Q_p is orthogonal to Q_s. Kallol and Hossein [61] obtained the required condition for completely continous linear operator T to be orthogonal to another completely continous linear operator A in the sense of James. Also it was shown that if T is orthogonal to A and $0 \notin \sigma_{ap}(A)$ then $sup\{|(Tu, v)| = \|u\| = 1$ and $(Au, v) = 0\}$. It was proved that the complex scalar λ_0 is characterized by the fact that there exist $\{x_n\}, \|x_n\| = 1$ such that $((T - \lambda_o A)_{x_n}, Ax_n) \to 0$ and $\|(T - \lambda_0 A)_{x_n}\| \to \|T - \lambda_0 A\|$.

Dragomir and Kikianty [29] introduced types of orthogonality of 2-HH norms and the characteristics for those orthogonalities were determined. Inner product spaces and strictly convex spaces were also characterized. Khalil and Alkhawaida [64] presented two new definition of orthogonality types. One was related to proximity in Banach spaces and other related to contractive projections. The relation between the two types was studied and basic properties of each type were presented. The reflection of such orthogonalities to compact operators was discussed. Khalil and Alkhawaida [64] introduced new definitions of orthogonality using theory of best approximation in Banach spaces and projections on subspaces in Banach spaces. Main properties and consequences of these definitions were studied. The relations to compact operators was also studied.

Hossein [47] extended the concept of orthogonality to Banach spaces. Completely continuous operators on Banach spaces that has orthonormal countable basis were also characterized. Cuixia and Senlin [24] studied homogeneity in normed linear spaces of isosceles orthogonality and that was an important notion of orthogonality from the two view point. Cuixia and Senlin [24] related homogeneous isosceles orthogonality to other types of orthogonality which include vectors with isometric reflection and vectors with l_2-summand and it was shown that a Banach space Z is a Hilbert space provided that the interior of the group of isosceles orthogonality with homogeneity property in the unit sphere of Z is nonempty. Moreover, a constant NH_Z that is geometric to determine isosceles orthogonality is non homogeinous was introduced. It was shown that $0 \leq NH \leq 2$ $NH_Z = 0$ provided Z is a Hilbert space and $NH_Z = 2$ given that Z is not a square uniformly.

Salah and Hacene minimized the C_∞-norm from $L(H)$ to C_∞ of suitable affine mappings through convex and differentiable analysis as studied in operator theory. The mappings considered generally elementary operator especially the generalized derivations that were the most important. As a consequence, global minima in terms of orthogonality was characterized in Banach spaces. Later Ionica [50] related Birkhoff orthogonality to notions in convex analysis. Hence, Ionica [50] obtained the Blanco and Turnsek results regarding the linear transformations in terms of Birkhoff orthogonality.

Ali Zamani and Mohammad [9] gave results on approximate Roberts orthogonality and approximate Birkhoff orthogonality and the properties of approximate Roberts orthogonality were also studied. Moreover, the set of mappings that preserve approximate Roberts orthogonality of type $\varepsilon \perp R$. It was shown that an ε-isometric scalar multiple is a mapping that preserves approximate Roberts orthogonality. Justyna [54] showed how different types of orthogonality have been described in functional equations. Justyna [54] introduced aspects of orthogonality, functional equations examples were given for vectors that are orthogonal. Some of their results and some applications were shown. Then, the factors affecting stability of some of functional equations were discussed considering different notions. Also, Justyna [54] mentioned the orthogonality equation and the challenge that preserve orthogonality. Finally, some open problems regarding those topics were stated.

Pawel [74] introduced an approximate and exact orthogonality relation and considered algebra of linear mappings that preserve approximate orthogonality. Pawel [74] studied the property of a linear mapping reserving the B-orthogonality and it was proved to be equivalent to the p, p_+-orthogonality (although these orthogonalities need not be equivalent). However, it was shown that every map that is linear with approximate orthogonality is a isometric scalar multiple. It was shown that a linear map which preserve Birkhoff-James orthogonality is a isometric scalar multiple. Pawel [74] gave some characterizations of linear mappings with approximate orthogonality in real normed spaces.

Later in [75] Pawel extended this study and showed that semi-orthogonality and p_+-orthogonality are not comparable unless it is for a smooth normed space. Consequently smooth spaces were characterized in terms of approximate orthogonality. In [10] Ali Zamani and Mohammad introduced the notion of approximate Roberts orthogonality set and investigated the properties of the given sets. To add, Ali Zamani and Mohammad [10] introduced the concept of approximate a-isosceles orthogonality and considered a group of transformations with approximate a-isosceles orthogonality.

Chaoqun and Fangyan [25] investigated maps between normed spaces with the orthogonality given by the norm derivative. Those maps were proved to be an isometric scalar multiple. Bhuwan [20] studied two new types of orthogonality from generalized carlsson orthogonality and some properties of orthogonality in Banach spaces were verified as Best implied Birkhoff orthogonality and Birkhoff orthogonality implied Best approximation. It was also shown that Pythagorean orthogonality implies Best approximation.

In [18] Balestro, Horst and Teixera introduced new geometric constants that differentiates Roberts orthogonality and Birkhoff orthogonality in spaces that are normed by characterizing Roberts orthogonality in two different ways through bisectors and using certain linear transformations. The main objective was to introduce new characterizations of Rorbert orthogonality. One of them was related to segments whose bisectors contain lines, and the other one associated this type of orthogonality to certain symmetries of the unit circle. Balestro studied geometrical structure of bisectors in normal planes and defined constant C_s, which quantifies the maximum symmetry of the unit circle regarding directions which are Birkhoff orthogonal.

From a geometric point of view [33] Debmalya, Kallol and Arpita studied two types of approximate Birkhoff-James othogonality in a space that is normed, and characterized them in the sense of normal cones. The concept of normal cones was characterized and related to approximate Birkhoff-James orthogonality in a Banach space of dimension 2 was explored. Uniqueness theorem was obtained for approximate Birkhoff-James orthogonality in a normed space. Their main aim was to study two different approximation of Birkhoff-James orthogonality, to have a good understanding of the properties of normed spaces. Among other things Debmalya, Kallol and Arpita [33] exhibited that the two types of approximate Birkhoff-James orthogonality have a close connection with normal cones in a normed space. Thomas [91] combined functional analytic and geometric view points on approximate Birkhoff orthogonality in generalized minkowskis spaces which are finite dimensional vector spaces endowed with a gauge. That was the first approach in those spaces.

In a normed space X, Ghosh, Debmalya and Kallol [44] related strict convexity to orthogonality of operators in terms of Birkhoff-James in $K(X)$, for completely continuous operators on X. It was shown that a Banach space that is reflexive Z is convex if given that $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} Q$ or $Rz = 0$ for some $x \in S_z$ with $\|Qz\| = \|Q\|$, $Q \in K(Z)$. It was shown that given Z is a Hilbert space of infinite dimension then for every $R \in L(Z)$ $R \perp_B Q \Rightarrow Q \perp_B R$ if Q is the zero operator. It was then proved that $R \perp_B Q \Rightarrow Q \perp_B R$ for a real Hilbert space Z, $Q \perp_B R \Rightarrow R \perp_B Q$ for every $R \in L(Z)$ if Q is the zero operator.

Debmalya [30] studied Birkhoff-James orthogonality defined on a real Banach space of finite dimension for bounded linear operators. The main reason for the study was in two ways, to determine Birkhoff-James orthogonality of transformations on a real Banach space whose dimension is finite and to characterize the symmetric properties of Birkhoff-James orthogonality of transformations defined on Z. Considering the obtained results, Debmalya [30] studied the left symmetric properties of Birkhoff-James orthogonality of mappings defined on $L(l_p^2)$ $(p \geq 2$. Letting $F, \|.\|$ to be a Banach space whose dimension is finite and $G_F = f \in F : \|f\| \leq 1$ and $G_F = f \in F : \|f\| = 1$ to be the unit ball and the unit sphere of the Banach space defined by the usual operator norm respectively.

Debmalya [30] introduced a particular notion motivated by geometric observations to determine Birkhoff-James orthogonality of vector space homomorphism for j, k in a vector space Z of which a norm is defined on a real Banach space of finite dimension, $k \in Z^+$ if $\|j+\lambda k\| = \|j\|$ for every $\lambda \geq 0$, also $k \in Z^-$ if $\|j + \lambda k\| = \|j\|$ for every $\lambda \geq 0$. The symmetric property of Birkhoff-James orthogonality of linear transformations on a real complete vector space Z on which a norm of finite dimension is defined was considered. So, Debmalya [30] considered this property in Banach spaces and proved some results similar to the symmetric property of of linear transformations on a real complete vector space Z on which a norm of finite dimension is defined. It was shown that there is nonzero liner operators $Q \in L(Z)$ such that Q is left symmetric in $L(Z)$. Lastly, using some of the results obtained, Debmalya [30] proved that $Q \in L_r^2$ $r \geq 2$, $r \neq 0$ is left symmetric given that Q is the zero operator. It was proved that $Q \in L(l_r^2)$ $(r \geq 2, r \neq \infty)$ is left symmetric in relation to Birkhoff-James orthogonality given that Q is the zero operator. Debmalya [30] concluded that the result holds for a strictly convex of any finite dimension and smooth real Banach spaces l_r^n $(r > 2, r \neq \infty)$.

Jacek [51] considered a linear operator $Q : Z \to Z$ on a normed space Z reversing orthogonality. That is, satisfying the condition $j \perp k \to Q_k \perp Q_j$ for all $j, k \in Z$ where $\perp$ stands for Birkhoff orthogonality. Kallol and Debmalya [60] studied Birkhoff James of two linear transformations Q, G on $(\mathbb{R}^n, \|.\|_\infty)$. Kallol and Debmalya [60] found the required property for Q to be orthogonal to G in terms of Birkhoff -James with some properties on Q. In [60] a condition necessary for the existence of two operators Q, G on $(\mathbb{R}^n, \|.\|_\infty)$ with $Q \perp_B G$ such that $z \notin \mathbb{R}^n$ with $\|z\|_\infty = 1$, $Q_z \perp_B G_z$ and $\|Q_z\|_\infty = \|Q\|_\infty$ was given. Kallol and Debmalya [60] found a condition on Q so that if $Q_z \perp_B G_z$ then there is $z \in \mathbb{R}^n$ with $\|z\|_\infty = 1$ such that $Q_z \perp_B G_z$ and $\|Q_z\|_\infty = \|Q\|_\infty$. Then, orthogonality of vectors in $(\mathbb{R}^n, \|.\|_\infty)$ was related to orthogonality of operators on $(\mathbb{R}^n, \|.\|_\infty)$.

In [42] Ghosh, Kallol and Debmalya studied the orthogonality of bounded linear transformations on $(\mathbb{R}^n, \|.\|_1)$ in the sense of Birkhoff-James. It had been shown that $Q \perp_B G \Rightarrow G \perp_B Q$ for all operators G on $(\mathbb{R}^n, \|.\|_1)$ given that Q obtains a norm at the extreme point, image that is left symmetric point of $(\mathbb{R}^n, \|.\|_1)$ and images of points that are extreme are zero. It was also shown that $G \perp_B Q \Rightarrow Q \perp_B G$ for all operators G provided that Q obtains norm the extreme points given that the images of extreme points are scalar multiples of extreme points. A necessary condition was obtained for an operator Q to be left symmetric. It was proved that $Q = q_{ij}$ is right symmetric given that for every $i \in \{1, 2, ...\}$ exactly one term $q_{i1}, q_{i2}...q_{in}$ is non-zero and of the same magnitude proved that Q is a left symmetric provided Q is the zero operator when the dimension is more than two. It was then shown that if Q is a transformation $(\mathbb{R}^2, \|.\|_1)$ then Q is left symmetric given that Q attains norms at only one extreme point say e, Q_e is symmetric and the other extreme point is zero.

In [43] Ghosh, Debmalya and Kallol studied Birkhoff James orthogonality of bounded linear transformations $(\mathbb{R}^n, \|.\|_\infty)$ and characterized the right and left symmetric operators on $(\mathbb{R}^n, \|.\|_\infty)$. In [57] Kallol, Debmalya and Arpita characterized the notion of approximate Birkhoff-James orthogonality for linear transformations on a normed space. Birkhoff James orthogonality on Hilbert space of either finite or infinite dimension was characterized in bounded linear transformations space and this improved the recent result by Chiemlink in which Birkhoff-James orthogonality of linear transformations on Hilbert space of finite dimension was characterized and also completely continuous operators on Hilbert space of finite dimension and also operators that are compact on any Hilbert space.

Kallol, Debmalya and Arpita [57] characterized Birkhoff-James orthogonality $Q \perp_B^\epsilon G$ acting on either finite or infinite dimension, approximate Birkhoff-James orthogonality $Q \perp_B^\epsilon G$ was determined in both cases where Q, G denotes completely continuous operators on a Banach space that is reflexive and the bounded linear operators Q, G on a vector space where a norm is defined of either finite or infinite dimension was also provided. Interrelation between the space ground and the space of bounded linear transformations was explored and $Q \perp_B^\epsilon G$ was characterized for $Q, G \in (XY)$ in the sense of norm attainment set M_Q where the space X is a Banach space that is reflexive. Kallol, Debmalya and Arpita [57] provided an alternative proof for the theorem which stated that in real normed space of finite dimension $Q \perp G$ provided there exist $j, k \in M_Q$ such that $G_j \in (Q_k)^+$ and $G_k \in (Q_k)^-$.

In [31] Debmalya, Kallol and Arpita studied Birkoff-James orthogonality of linear mappings that are bounded and characterized linear mappings on a real space of infinite dimension that is normed through Birkhoff James. While Birkhoff-James orthogonality was characterized for bounded linear operators defined on a Hilbert space or a finite dimensional Banach space, the problem of of characterizing Birkhoff-James orthogonality on normed linear spaces of infinite dimension for linear mappings that are bounded remained unsolved. Motivated by the result on rotund bounded linear mappings, Birkhoff-James orthogonality of rotund points in the space of bounded linear operators was obtained. In order to obtain the desired characterization for rotund points and for general bounded linear operators.

Debmalya, Kallol and Arpita [31] introduced a new definition which was essentially geometric in nature and hence in this manner a Birkhoff-James orthogonality for linear operators that are bounded of general normed spaces was characterized. ϵ-orthogonality was decomposed to completely characterize bounded linear mappings that are bounded through Birkhoff-James. As a consequence, Birkhoff James orthogonality on a real normed linear space for linear functionals that are bounded was characterized provided the dual space is strictly convex. In [31] required conditions for smoothness of linear that are bounded on a normed linear space of infinite dimension was provided. In [58] Kallol, Arpita and Pawel studied left symmetric mappings defined on a Banach space of infinite dimension in the sense of Birkhoff-James.

In [62] Kallol, Debmalya, Arpita and Kalidas studied Birkhoff-James orthogonality of bounded linear mappings on complex complete vectors spaces on which a norm is defined and obtained a complete characterization of the same. As a way of obtaining new definitions, it was illustrated that there is a possibility in spaces that are complex, to introduce orthogonality of linear mappings similar to the real paces. It was shown that, operator theoretic characterization of Birkhoff-James orthogonality in the real case could be obtained in form of corollaries to their recent study. In fact, Birkhoff-James orthogonality of completely continuous operators was characterized in the complex form in order to differentiate the complex form from the real one. The left symmetric linear operators on complex two-dimensional l_p space if and only if J is the zero operator was also studied. In [77] Sanati and Kardel characterized the class of operators that preserve orthogonality on Hilbert space H of infinite dimension as a scalar multiple of unitary operators of H and the subspaces of H that are closed. For an orthogonal preserving operator, it was shown that the spectrum is any circle that is centred at the origin.

Debmalya, Kallol, and Arpita [32] studied Birkhoff-James orthogonality for vector spaces in which a norm is defined for completely continuous operators. Their main aim was to determine Birkhoff-James orthogonality of completely continuous operators defined on a normed linear space. Through the concept of semi-inner-products and the similar ideas in normed spaces, some of the recent results were generalized and improved. In particular, Euclidean spaces was characterized and it was also proved that there is a possibility of retrieving the norm of a completely continuous operator in the terms of Birkhoff-James orthogonality. Certain results of approximation type were also presented in the space of operators that are bounded.

Debmalya, Kallol, and Arpita [32] introduced the Bhatia-Semrl theorem for operators that are compact on a Hilbert space on infinite dimension and also characterized Euclidean spaces for all Banach spaces of finite dimension. The concept of inner product spaces was correlated with the types of r^+ and r^-. This enabled them to get the norm of a completely continuous linear operators in relation to its interaction with Birkhoff-James orthogonality. Finally, few approximation results were presented in Hilbert spaces and Banach spaces

In [68] Kallol presented results on Birkhoff-James and smoothness of operators in normed spaces. Kallol [68] explored the orthogonality relation between elements in Banach spaces Z of operators $L(Z)$ that are linear and bounded. Smoothness of the space of operators that are linear and bounded was also studied. In [21] Bhuwan and Prakash applied orthogonality in the best approximation in normed linear spaces . Hence, it was shown that Birkhoff orthogonality means best approximation and best approximation means Birkhoff orthogonality. It was shown that for ε-orthogonality, ε-best approximation means ε-orthogonality. In [21] Bhuwan and Prakash showed how pythagorean orthogonality and best approximation, isosceles orthogonality and ε-best approximation are related in normed spaces.

In [11] Ali Zamani generalized operators for a semi-inner product on a Hilbert space in the sense of Birkhoff-James. Given that P and Q are linear transformations on a complex Hilbert space Z, the relation $P \perp_J^K Q$ was defined if P and Q are bounded with a semi-norm endowed with a positive operator J that satisfy $\|P + \gamma Q\|_J \geq \|P\|_J$ for a complex γ. Zamani [11] proved that $P \perp_J^K Q$ given that there exist a sequence $\{z_n\}$ with a norm of 1 in Z such that $\lim_{n\to\infty} \|Pz_n\|_J = \|P\|_J$ and $\lim\langle Pz_n, Qz_n\rangle_J = 0$. Some distance formulas in Semi-Hilbert spaces were also Provided.

In [12] Birkhoff-James orthogonality for linear transformations was characterized and proved to be a vector space of operators on arbitrary Banach spaces. Arbitrary Banach spaces were characterized and some conditions were obtained. Arpita and Kallol [12] studied orthogonality in space of operators $L(Z)$ on arbitrary Hilbert space Z, both in relation to operator norm and numerical radius norm.Orthogonality of Birkhoff-James orthogonality for operators in spaces that are completely normed was also obtained. Their main goal was to determine Birkhoff-James orthogonality of the operator $T \in L(Z, W)$ to the subspace of $L(Z, W)$ in an arbitrary Banach spaces Z and W set up.

Arpita and Kallol [12] first characterized $Q \perp R$ where $Q \in L(W, Z)$ and R is a subspace of $L(W, Z)$ of finite dimension and W is a reflexive Banach space given that Z is a Banach space of finite dimension. For arbitrary Banach spaces W and Z of $L(W, Z)$ and for an arbitrary subspace W, $Q \perp_B R$ under suitable conditions. Arpita and Kallol [12] also characterized $T \in L(W, Z)$ to a subspace of $L(H)$ in the sense of Birkhoff-James on a Hilbert space H of infinite dimension. Later, it was discovered that in order to characterize orthogonality of operators, there was need for the operators to attain norms. In [88] Bottazi, Conde and Debmalya determined the orthogonalities of Birkhoff-James and isosceles for operators defined on Hilbert spaces and Banach spaces. There was no other universal concepts of orthogonality in a Banach space unlike in Hilbert spaces. Then, it was found that there is a possibility of having several types of orthogonality in such a space, in which each characterizes certain particular concept of orthogonality in Hilbert spaces. Since lack of a standard orthogonality led to the differences of Hilbert spaces and Banach spaces, Bottazi, Conde and Debmalya [88] explored linear operators in terms of Birkhoff-James in a different aspect and discussed some applications to this regard. A study on Isosceles orthogonality of mappings that are linear and bounded on a Hilbert space was done and related properties were determined, and properties of disjoint support were also included. It was shown that for bounded linear operators between Banach space of infinite dimension, Bhatia-Semrl theorem verbatim of finite dimension was extended under some additional assumptions.

Bottazi, Conde and Debmalya [88] studied the properties of the set $O_{P,A} = \{x \in S_X : P_x \perp P_y\}$ for any $P \in L(X, Y)$ and characterized the Hilbert space of finite-dimension in relation to the new introduced concept. Bottazi, Conde and Debmalya [88] focused on orthogonality of operators that were positive and those that were linear defined on a Hilbert space. Isosceles orthogonality was generalized for two positive bounded linear operators and some remarks between Birkhoff-James orthogonality and Isosceles orthogonality were discussed. Properties of Isosceles orthogonality and Birkhoff orthogonality were further explored in Banach spaces. Bottazi, Conde and Debmalya [88] concluded by establishing that Rorbert's orthogonality is more agreeable than that of either Birkhoff-James and Isosceles orthogonality.

In [22] Bhuwan and Prakash enlisted properties of Birkhoff-Orthogonality and Carlsson orthogonality along with it, Bhuwan and Prakash [22] introduced two new particular cases of Carlsson orthogonality and checked some properties of orthogonality in relation to these particular cases in normed spaces. Bhuwan and Prakash [22] showed how isosceles, Rorbert and Pythagorean orthogonalities can be derived from the carlsson orthogonality and obtained two new orthogonality relations for the Carlsson.

In [73] Priyanka and Sushil gave the known properties of Birkhoff-James orthogonality in Banach space. Concepts of orthogonality, the Gateaux derivative and the sub-differential set of function of norm were discussed and important distance formulas that were determined by characterizing Birkhoff-James orthogonality. Priyanka and Sushil [73] mentioned the relation between orthogonality and properties that are geometric for spaces that are normed. This lead to the determination of different related concepts like characterization of smooth points and extreme points, sub differential sets and ψ-Gateaux derivertive sets. Priyanka and Sushil [73] also characterize symetric property of orthogonality. Generalisations of orthogonality in different Banach spaces were detemined together with their applications. The characterizations obtained were used to determine distance formulas in certain Banach spaces.

In [87] Tanaka and Debmalya characterized the left and right symmetric points in the terms of Birkhoff orthogonality in $L(G, R)$ and $K(G, R)$ where G, R are complex Hilbert space and $L(G, R)$ ($K(G, R)$) is the space of all compact bounded mappings from G into R. Their main aim was to improve the notion of local symmetry for a strong type of Birkhoff orthogonality. It was shown that an element J in $L(G, R)$ ($K(G, R)$) is left symmetric for $\perp_{L(G)}$ ($\perp_{K(G)}$) in $L(G, R)$ ($K(G, R)$) provided that J is rank one operator, it turned out that $J \in L(G, R)$ given that:

(i). J is right invertible such that Q is of infinite dimension or $dimG > dimR$.

(ii). J is an isometric scalar multiple where G is of infinite dimension and $dimG < dimR$ while $J \in K(G, R)$ is right symmetric for $\perp_{K(G)} \in K(G, R)$ if J has the dense range.

Debmalya, Ray and Kallol [35] explored the relation between the orthogonality of bounded linear operators and the elements in the ground space. It was shown that if $Q, R \in L(W, Z)$ satisfy $Q \perp_B R$, and that there exists $w \in W$ so that $Q_w \perp R_z$ with $\|z\| = 1, \|Q_W\| = \|T\|$, given that W, Z are normed linear spaces. The concept of property D_n for a Banach space was introduced and its relation with orthogonality of operators on Banach spaces was illustrated. Debmalya, Ray and Kallol [35] further studied the property D_n for various polyhendra Banach spaces. Their aim was to study Bhatia-Semri(BS) property in polyhedral Banach spaces for bounded linear operators.

For orthogonality of elementary operators, orthogonality of range and kernel of normal derivations was determined by Anderson [5] . In the study, Anderson [5] showed that if J and P are operators in $L(Z)$ such that J is normal and $JP = PJ$ then for every $Y \in L(Z)$,
$\|\delta_J(Y) + P\| \geq \|P\|$ where $\|.\|$ is the usual operator norm. Anderson [5] showed that if Q is isometric or is normal then the range of δ_Q is orthogonal to its nullspace. Also Anderson [5] proved that if Q is normal and has infinite number of points then the closed linear space of the range and null space of δ_Q is not all of $L(Z)$. Kittaneh [65] extended the study and showed that given J and P are operators in $L(Z)$ such than J is normal, P is a Hilbert Schmidt operator and $P \in \{J\}$ then for all $Y \in L(Z)$, $\|\delta_J(Y) + P\|_2^2 \geq \|\delta_J(Y)\|_2^2 + \|P\|_2^2$ where $\|.\|_2$ is the Hilbert Schmidt operator norm. Therefore, the range of δ_J if orthogonal to the kernel of δ_J for the Hilbert Schmidt operators in the usual sense.

In the schatten p-norms Kittaneh [66] used the Gateaux differentiability and the usual operator norm to determine the range and kernel orthogonality of elementary operators in relation to p-norms. In [37] Duggal considered an elementary operator δ_{ab} in which the operators a, b, x are hyponormal, the operators a_1, b_2 are normal and a_1 commutes with b_2.

In [89] Turnsek studied the elementary $\phi; L(Z) \rightarrow L(Z)$ defined by $\phi(V) = \sum_{i=1}^{k} A_i V B_i$ and $\phi^*(V) = \sum_{i=1}^{k} A_i^* B_i^*$. Tursek [89] proved that

(i). When $\phi \leq 1$, then $\|\phi(V) - V + Q\| \geq \|Q\|$ for every $V \in L(Z)$ and $Q \in Ker\phi$.

(ii). When $\sum_{i=1}^{k} A_i A_i^* \leq 1$, $\sum_{i=1}^{k} A_i^* A_i \leq 1$, $\sum_{i=1}^{k} B_i B_i^* \leq 1$ and

$\sum_{i=1}^{k} B_i^* B_i \leq 1$ then, for $Q \in Ker\phi \cap Ker\phi^* \cap l_p$ $\|\phi V - V + Q\|_p \geq \|Q\|_p$ and $\|\phi^* V - V + Q\|_p \geq \|Q\|_p$ for every $V \in L(Z)$.

(iii). $(R_i)_{i=1}^{k}$ and $(U_i)_{i=1}^{k}$ be sequences of normal operators that commute separately and let $\delta(Z) = \sum_{i=1}^{k} R_i V U_i$. If $\delta(Z) \in l_2$ and $Q \in ker\delta \cap l_2$, then $\|\delta(Z) + Q\|_2^2 = \|\delta(Z)\|_2^2 + \|Q\|_2^2$.

Turnsek [89] considered a normed algebra A and a linear operator $\phi : A \to A$ and proved that the range $\phi - 1$ is orthogonal to its kernel if $\|\phi\| \leq 1$. This could also be applied to the case when $\phi; L(Z) \to L(Z)$ is an arbitrary elementary operator defined by $\phi(Z) = \sum_{i=1}^{k} A_i Z B_i$.

Dragoljub [27] proved the orthogonality of an important elementary operator in relation to the unitary invariant norms and their association with the norm ideals of operators. The group consisted the mapping $Q : L(Z) \to L(Z)$, $Q(V) : FVH + JVP$ where $L(Z)$ denotes the group of all bounded operators and F, H, J and P are normal operators so that $FJ = JF$, $HP = PH$ and $KerF \cap KerJ = KerH \cap KerP = \{0\}$. Dragoljub [27] established this set in sense in which an orthogonality result holds.

Bachir and Hashem [17] presented a new class of finite operators and extended orthogonality results to some finite operators. In [17] some commutativity results were also generalized. Their main goal was to investigate the orthogonality of Ran$\delta_{A,B}$ and $Ker\delta_{A,B}$ for certain finite operators. It was proved that $Ran(\delta_{A,B})$ is orthogonal to $Ker(\delta_{A,B})$ where A is dominant and B^* is M-hyponormal. Duggal and Harste [38] studied orthogonality and properties of range closure for some elementary operators as proved for hyponormal operators or contractions on Hilbert spaces.

Okelo and Agure [72] presented various types and aspects of orthogonality in spaces that are normed. In [72] the range and kernel orthogonality results for elementary operators were given and the operators that characterize them were then provided. In [13] Bouali and Bouhafsi exhibited pair (Q, R) of operators such that orthogonality of $\delta_{Q,R}$ is valid for the usual operator norm. Range and nullspace of $\delta_{Q,R}$ results were obtained in relation to the group of unitarily invariant norms.

Bachir and Nawal [16] studied and characterized the range-kernel orthogonality of the points $C_1(H)$, the trace class operators in nonsmoothness case and gave a counter example. In [70] Okelo characterized orthogonality of operators that are elementary in groups that attain norms. Okelo [70] gave conditions for operators to be norm attainable in Hilbert spaces. In [70] range-kernel orthogonality results were given for elementary operators in norm-attainable classes.

1.2 Basic concepts

This section has mathematical concepts that will be used throughout this note. In particular we define field, vector space, norm, Banach space, Hilbert space, inner product, and some operators among other terms.

Definition 1.1 (67, Section 1)**.** A field $\mathbb{F}$ is a non-empty set with two operations called multiplication and addition denoted by (.) and + such that the following axioms hold;

(i). both $p + r$ and $p.r$ are in $\mathbb{F}$, for all $p,\ r \in \mathbb{F}$.

(ii). $p + (r + i) = (p + r) + i$ and $p.(r.i) = (p.l).i$, for all $(p,\ r,\ i) \in \mathbb{F}$-associativity property.

(iii). Commutativity of addition and multiplication: $p + r = r + p$ and $p.r = r.p$, for all $p,\ r, \in \mathbb{F}$.

(iv). Existence of additive and multiplicative identity: there exist two different elements 0 and 1 *in* $\mathbb{K}$ such that $p + 0 = p$ and $p.1 = p$, for all $p \in \mathbb{F}$.

(v). Existence of additive inverses: for every $p \in \mathbb{F}$, there exist an element in $\mathbb{F}$, denoted by $-p$, called the additive inverse of p, such that $p + (-p) = 0$.

vi. Existence of multiplicative inverses: for every $p \neq 0$ in $\mathbb{F}$, there exist an element in $\mathbb{F}$ denoted by p^{-1} *or* $\frac{1}{p}$ called the multiplicative inverse of a, such that $p.p^{-1} = 1$.

vi. Distributivity of multiplication over addition: $p.(r + i) = (p.r) + (p.i)$, for every $p,\ r,\ i \in \mathbb{F}$.

Definition 1.2 (76, Section 1)**.** A vector space over a field $\mathbb{F}$ is a non-empty set Z with two binary operations, addition mapping $Z \times Z$ into Z and scalar multiplication mapping $\mathbb{F} \times Z$ into Z satisfying the following properties ;

(i). Closure of addition: $g + q = q + g \ \ for \ \ all \ \ g, q \in Z$.

(ii). Associativity of addition: $g+(q+h) = (g+q)+h \ \ for \ \ every \ \ g,\ q,\ h \in Z$.

(iii). There is an element denoted by 0 (called the zero vector) such that $g + 0 = 0 + g = g$, for all $g \in Z$.

(iv). For every $q \in Z$, there exist an element denoted by $-q$ such that $q + (-q) = (-q) + q = 0$.

(v). $1 \cdot g = g$ for all $g \in Z$ where I is the identity for $\mathbb{F}$.

(vi). $(\mu\eta)g = \mu(\eta g)$ for every $g \in Z$ and $\mu, \eta \in \mathbb{F}$.

(vii). Distributivity: $(\mu + \eta)g = \mu g + \eta g$, for every $g \in Z$ and $\mu, \eta \in \mathbb{F}$ and $\mu(g + q) = \mu g + \mu q$, for all $g,\ q \in Z$ and $\mu, \eta \in \mathbb{F}$.

Example 1.3 (7, example 1)**.** Let $Z = \mathbb{R}^n$ be a group of all real numbers. This is a linear space over the reals. We have $(i_1, ..., i_n) + (d_1, ..., d_n) = (i_1 + d_1, ..., i_n + d_n)$ and $a(i_1, ..., i_n) = (ai_1, ..., ai_n)$.

Example 1.4 (67, example 3)**.** Suppose $Z = \mathbb{C}$ be the group of all complex numbers. This is a linear space over complex numbers. Addition and scalar multiplication are defined as in the example above.

Example 1.5 (76, example 1)**.** Suppose Z be a collection of all sequences that are not finite $(i_1, i_2, ...)$ of real numbers with addition being coordinate-wise, that is $(i_1, i_2...) + (d_1, d_2...) = (i_1 + d_1, i_2 + d_2, ...)$ and similarly for scalar multiplication.

Example 1.6 (7, example 2)**.** If H is a set and let Z be the collection of real valued bounded functions on H. We define $i + z$ by $(i + z)(s) = i(s) + z(s)$ for each $s \in H$ and ai by $(ai)(s) = ai(s)$ for each $s \in H$.

Definition 1.7 (8, Definition 3.0)**.** Let Z be a non empty set. A function $d : Z \times Z \to \mathbb{R}$ is a metric on Z if the following properties are satisfied:

(i). $d(n, q) \geq 0$ and $d(n, q) = 0$, if and only if $n = q$ for every $n, q \in Z$.

(ii). $d(n, q) = d(q, n)$.

(iii). $d(n, h) \leq d(n, q) + d(q, h)$

The ordered pair (Z, d) is called a metric space.

Definition 1.8 (8, Definition 3.1). Let X be a linear space over $\mathbb{F}$ .Then a norm on X is a non-negative real-valued function $\|.\| : X \to \mathbb{R}$ such that $\forall\ w, z \in X$ and $\eta \in \mathbb{F}$ the following properties are satisfied:

(i). $\|w\| \geq 0$ and $\|w\| = 0$, if and only if $w = 0$.

(ii). $\|\eta w\| = |\eta|\|w\|$.

(iii). $\|w + z\| \leq \|w\| + \|z\|$

The ordered pair $(X, \|.\|)$ is called a normed space.

Definition 1.9 (67, Definition 3.5). If Z is a vector space with norm $\|.\|$ and $d : Z \times Z \to \mathbb{R}$ is a metric defined by $d(w.z) = \|w - z\|$, then d is called the metric associated with the norm.

Definition 1.10 (76, Definition 3.15). A Banach space is a complete normed linear space.

Definition 1.11 (8, Definition 3.18). Let Z be a real or complex vector space. An inner product on Z is a function $\langle ., . \rangle : Z \times Z \to \mathbb{J}$ such that $\forall\ w,\ z,\ k \in Z$ and $\lambda, \beta \in \mathbb{J}$; if it satisfy:

(i). $\langle w, w \rangle \geq 0$ and $\langle w, w \rangle = 0$, if and only if $w = 0$.

(ii). $\langle \alpha w + \beta z, k\rangle = \alpha\langle w, k\rangle + \beta\langle z, k\rangle$.

(iii). $\langle \lambda w, z\rangle = \lambda\langle w, z\rangle$.

(iv). $\langle w, z\rangle = \overline{\langle z, w\rangle}$.

The ordered pair $(X, \langle ., .\rangle)$ is called an inner product space.

Example 1.12 (67, example 2)**.** Let $X = \mathbb{F}^n$ for $w = (w_1...w_n)$ and $z = (z_1...z_n)$ in X define $\langle w, z\rangle = \sum_{i=1}^{n} w_i\overline{z_i}$.

Example 1.13 (8, example 1)**.** Let $X = l_0$ the space of sequences of real or complex numbers that are finitely non-zero. For $w = (w_1...w_n)$ and $z = (z_1...z_n)$ in X define $\langle w, z\rangle = \sum_{i=1}^{\infty} w_i\overline{z_i}$.

Example 1.14 (6, example 4)**.** Let $Z = l_2$ the space of all sequences $w = (w_1, w_2...)$ of real or complex numbers for $\sum_{i=1}^{\infty} |w_i|^2 < \infty$. For $w = (w_1...w_n)$ and $z = (z_1...z_n)$ in X define $\langle w, z\rangle = \sum_{i=1}^{\infty} w_i\overline{z_i}$.

Example 1.15 (7, example 1)**.** Let $Z = C[q, s]$ the space of all continuous complex valued function on $C[q, g]$ for $q, g \in Z$ define $\langle q, g\rangle = \int_q^g q_t\overline{g_t}dt$.

Definition 1.16 (67, Definition 3.21)**.** Suppose Z is a real or complex valued vector space with an inner product $\langle ., .\rangle$. Then X is an inner product space.

Definition 1.17 (76, Definition 3.26)**.** A Hilbert space H is a complete inner product space.

Remark 1.18. Any Hilbert space is a Banach space, but the converse is not necessarily true.

Example 1.19. $\mathbb{F}$ with the standard inner product is a Hilbert.

Definition 1.20 (6, Definition 3.6-1)**.** An operator P is said to be linear if, for every pair of vectors w and z and scalar λ, $P(l+d) = P(l) + P(d)$ and $P(\lambda l) = \lambda P(l)$.

Definition 1.21 (67, Definition 3.2-1)**.** Two vectors $w,\ z \in H$ are called orthogonal, denoted by $w \perp z$ if $\langle w, z \rangle = 0$.

Definition 1.22 (70, Section 1)**.** Consider a normed space $\mathcal{D}$ and let $T{:}\mathcal{D} \to \mathcal{D}$. T is said to be an elementary operator if it can be represented in the following form $T(X) = \Sigma_{i=1}^{n} S_i X P_i$ for all $X \in \mathcal{D}$ where S_i and P_i are fixed in $\mathcal{D}$.

Example 1.23. Let $S = L(Z)$ for $S, P \in L(Z)$ we define particular elementary operator.

(i). The left multiplication operator $L_S : L(Z) \to L(Z)$ by $L_S(X) = SX, \forall\ X \in L(Z)$.

(ii). The right multiplication operator$R_P : L(Z) \to L(Z)$ by $R_P(X) = XP, \forall\ X \in L(Z)$.

(iii). The generalized derivation by $\delta_{S,P} = L_S - R_P$.

(iv). The basic elementary operator by $M_{S,P}(X) = SXP, \forall\ X \in L(Z)$.

(iv). The Jordan elementary operator by $\mu_{S,P}(X) = SXP + PXS, \forall\ X \in L(Z)$.

Definition 1.24 (72, Section 1)**.** The range of an operator $P : L(H) \to L(H)$ is defined as $Ran(T) = \{y \in L(H) : y = T(x)\ \forall\ x \in L(H)\}$.

Definition 1.25 (72, Section 1)**.** The kernel of an operator $T : L(H) \to L(H)$ is defined as $Ker(T) = \{x \in L(H) : T(x) = 0\ \forall\ x \in L(H)\}$.

Definition 1.26 (92, Section 2)**.** A bounded linear operator S on a Hilbert space H is called finite if $\|I - SX - XS\| \geq 1$ for each $X \in L(H)$.

Definition 1.27 (13, Section 2)**.** A proper two sided ideal J in $L(H)$ is called a norm ideal if there is a norm on J possessing the following properties:

(i) $(J, |\|.\||)$ is a Banach space.

(i) $|\|SVP\|| \leq \|S\||\|V\||\|P\|$ for every $S, P \in L(H)$ and for every $V \in J$.

(i) $|\|V\|| = \|V\|$ for V a rank one operator.

1.3 Statement of the problem

Let Ω be a normed space and consider an elementary operator T on Ω. Various notions of orthogonality conditions have been obtained for elementary operators on normed spaces. However, orthogonality conditions have not been obtained when the elementary operators are finite in complex spaces. In this study therefore we considered finiteness of elementary operators and established orthogonality conditions for these operators in terms of James-Birkhoff orthogonality when the normed spaces are complex.

1.4 Objectives of the study

The objectives of the study are to:

(i). Characterize finiteness of elementary operators in complex normed spaces.

(ii). Establish orthogonality conditions for finite elementary operators in complex normed spaces.

(iii). Determine Birkhoff-James orthogonality for finite elementary operators in complex normed spaces.

1.5 Significance of the study

Orthogonality in inner product spaces is a binary relation that can be expressed in many ways without necessarily mentioning the inner product space. In normed spaces, great part of such definitions have also sense. This simple observation is at the base of many notions of orthogonality in these more general structures. The results obtained from this study are useful in quantum theory in estimation of the distance between the identity operator and the commutators.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

Orthogonality is linked with uniform convexity, strict convexity and smoothness of the space which are the most important geometric characteristics of normed linear spaces. In this chapter, we have reviewed literature related to finite operators, orthogonality in normed spaces and orthogonality of elementary operators.

2.2 Finite operators

A bounded linear operator T on a normed space Ω is called finite if $\|I - TX - XT\| \geq 1$ for each $X \in L(\Omega)$. Williams [92] showed that the group of finite operators involves operators that are normal, is uniformly closed and involves operators with a completely continuous direct summand, and every Banach algebra with an involution satisfying properties of adjoint characterized by every member. The results implied that the group of operators with a reducing subspace of finite dimension is non-uniformly dense. It was then proved that the group of self-commutators is uniformly closed. The following are some of William's results.

Theorem 2.1. *[92, Theorem 4] The following properties are equivalent on an operator T:*

(i). *T is finite.*

(ii). $\inf_x \|I - TX - XT\| = 1$.

(iii). *There exists $f \in \wp$ such that $f(TX) = f(XT)$ for every $X \in L(H)$.*

Theorem 2.1 shows that the class of finite operators involves every operator with a completely continuous direct summand and every normal operator but did not establish orthogonality conditions for finite operators.

Elalami [39] gave a new class of finite operators through the concept of the reducing approximate spectrum of an operator. In this case the concept of completely finite operators was introduced. Those are operators A such that A_E is finite for any orthogonal reducing subspace E of A. For those operators Elalami [39] gave characterizations and proved that dominant operators are completely finite. The following are Elalami's main results:

Proposition 2.2. *[39, Proposition 1.1] Let $S \in L(H)$. If $\delta_{ra}S$ is nonempty, then S is finite.*

Proposition 2.2 shows that an operator A is finite if the reducing approximate spectrum of A is not empty but but did not give a detailed description of finite operators.

Proposition 2.3. *[39, Proposition 2.1] If* $\|A\| = r(A)$ *then dist*$(A, R\delta_A) =$ $\|A\|$.

Proposition 2.3 shows that the range of normal derivation of A is orthogonal to it's kernel if the norm of A is equivalent to the $r(A)$ but but did not give a detailed description of finite operators.

Duggal [36] improved the Du Hong-Ke inequality to $\|QZQ - Z + G\| \geq \|Z\|$ for all operators Z. Indeed, Duggal [36] proved that Du Hong-Ke is equivalent to Anderson inequalities and it was shown that the inequality of Du Hong-Ke is valid for unitary norms that are invariant. The following are Duggal's main results.

Theorem 2.4. *[36, Theorem 1] Suppose* G *is an operator that is normal and that* $R(G, G^*)(S) = 0$ *for every* $S \in L(H)$ *then* $\|R(G, G^*) + S\| \geq \|S\|$ *for all* $X \in L(H)$

Theorem 2.4 Shows that the range of G and its adjoint is orthogonal to its null space if range of G and its adjoint is equivalent to zero but the study was limited to the characterization of finiteness of elementary operators.

Corollary 2.5. *[36, Corollary 1] If* Q *and* N *are normal operators such that* $R(Q, N)(S) = 0$ *for every* $S \in L(H)$*, then* $\|R(Q, N) + S\| \geq \|S\|$ *for all* $X \in L(H)$*.*

Corollary 2.5 considered normal operators and showed that the range of $\delta_{Q,N}$ is orthogonal to its null space but did not characterize finiteness of this elementary operators.

Corollary 2.6. *[36, Corollary 2] Suppose $Q, G \in L(H)$ are normal operators such that $C(Q,G)(S) = 0$ for some $S \in L(H)$, then $\|C(Q,G)+S\| \geq \|S\|$ for all $X \in L(H)$.*

Corollary 2.6 considered normal operators their orthogonality but did not characterize finiteness of this elementary operators.

In [86] Takayuki, Masatoshi and Takeaki introduced "class A" operators provided by the inequality of an operator which contained the group of paranormal operators and the group of log-hyponormal operators. It was discovered that their results consists the proof of Ando's results where every log-hyponormal operator is paranormal. New classes linked to class A operators and paranormal operators were also introduced. The following are some of their results:

Theorem 2.7. *[86, Theorem 2] 1. All log-hyponormal operators are class $A(k)$ operators. 2. All invertible class A operators are class $A(k)$ operators to $K \geq 1$.*

Theorem 2.7 shows that an invertible A operators are $A(k)$ operators and log-hyponormal operators are $A(k)$ operators and they are finite operators but did not give a detailed description of finite operators.

Salah [79] gave a set of finite operators of the form $S+G$ where $S \in L(H)$ and G is compact. Salah [79] proved that $w_o(\delta_{S,P}) = c_o\delta(\delta_{S,P})$, where $w_o(\delta_{S,P})$ is the the numerical range of $\delta_{S,P}$ and $c_o\delta(\delta_{S,P})$ is the the convex hull of $\delta(\delta_{S,P})$ for certain operators $S, P \in L(H)$, $\delta_{S,P}$ is the operator on $L(H)$ given by $\delta_{S,P} = SX - XP$, $X \in L(H)$. The following are salah's main results:

Theorem 2.8. *[79, Theorem 3] $F(H)$ involves the following operators.*

(i). $S \in L(H)$ such that $\delta w(S) \cap \delta(S) \neq 0$.

(ii). dominant operators.

Theorem 2.8 shows that dominant operators are finite but did not give a detailed description of finite operators.

Theorem 2.9. *[79, Theorem 3] $F(H)$ involves the following operators.*

(i). $\|Q\| = w(Q)$ where $w(Q)$ is the numerical radius of Q.

(ii). $Q \in L(H)$ such that Q satisfies $\mathbb{C}$.

Theorem 2.9 shows that if the norm of Q is equal to the numerical radius of Q then Q is finite but did not give a detailed description of finite operators.

Corollary 2.10. *[79, Corollary 9] $F(H)$ involves the following operators.*

(i). $Q = J + G$, G compact and J dominant.

(ii). $Q = J + G$, G compact and $J \in \mu$.

Corollary 2.10 shows that a compact operator G plus dominant operator J are finite but did not give a detailed description of finite operators.

In [81] Salah characterized the operators $Q \in L(H)$ that is orthogonal to the range of $ker\delta_{S,P}$ for operators which are not normal$S, P \in L(H)$ in the sense of James. The following are some of salah's main results.

Corollary 2.11. *[81, Corollary 2.2] Suppose* $S, P \in L(H)$ *such that* $S^m = I$ *and* $P^m = I$ *for some integer m. Then* $\|SX - XP - Q\| \geq \|Q\|$ *for all* $X \in L(H)$ *for all* $Q \in ker\delta_{S,P}$.

Corollary 2.11 considered normal operators and showed that the range of $\delta_{S,P}$ is orthogonal to its null space but did not characterize finiteness of these elementary operators.

Bachir [15] gave results on orthogonality log-hyponormal operators and dominant operators, then results on commutativity were obtained. The following are Bachir's main results:

Proposition 2.12. *[15, Proposition 3.1] Suppose S is dominant and that Q is a normal operator and that* $SQ = QS$*, then for all* $\lambda \in \delta_p(Q)$, $\|\lambda\| \leq dist(Q, R(\delta_S))$

Proposition 2.12 shows that dominant operators are finite but did not characterize finiteness of elementary operators.

Proposition 2.13. *[15, Proposition 3.3] Let S be dominant and Q be a normal operator and that* $SQ = QS$*, then for all* $\lambda \in \delta_p(Q)$, $\|\lambda\| \leq dist(Q, R(\delta_S))$

Proposition 2.13 showed that dominant operators are finite but did not characterize finiteness of these elementary operators.

Theorem 2.14. *[15, Theorem 3.4] Suppose S is dominant and* P^* *is p-hyponormal operator or log-hyponormal, then given* $Q \in ker(\delta_{S,P})$*, we have* $\|Q\| \leq dist(Q, R(\delta_{S,P}))$

Theorem 2.14 shows that dominant operators and log-hyponormal are finite but did not give a detailed description of finite operators.

In [80] Salah described paranormal operators and analyzed that they are finite and presented some other examples of operators that are finite. An extension of inequality $\|I - AX - XA\| \geq 1$ was also given. The following are some of salah's main results.

Theorem 2.15. *[80, Theorem 1.7]*

(i) A log-hyponormal operator is a class A operator.

(ii) A class A operator is a paranormal operator.

Theorem 2.15 shows that log-hyponormal operators, paranormal operators and class A operators are finite operators but did not establish orthogonality conditions for finite operators.

Corollary 2.16. *[80, Corollary 2.1] The following are finite operators.*

(i). Hyponormal operators,

(ii). p-Hyponormal operators,

(iii). Class A operators,

(iv). Log-hyponormal operators.

Corollary 2.16 shows that hyponormal, p-hyponormal, class A operators and log-hyponormal operator are finite but the study was limited to orthogonality conditions for finite operators.

Lemma 2.17. *[80, Lemma 2.1] If S is paranormal and if Q is a normal operator and that $SQ = QS$, then for all $\lambda \in \delta_P(Q)$, $|\lambda| \leq \|Q - (SX - XS)\|$ for all $X \in L(H)$.*

Lemma 2.17 shows that every paranormal operator is finite given that T is normal and $\lambda \in \delta_P(Q)$ but did not establish orthogonality conditions for finite elementary operators.

Theorem 2.18. *[80, Theorem 2.2] If G is paranormal, then for every normal operator Q such that $GQ = QG$, $\|Q - (GX - XG)\| \geq \|Q\|$ for all $X \in L(H)$.*

Theorem 2.18 shows that a paranormal operator is finite for every normal operator Q but did not establish orthogonality conditions for finite elementary operators.

Corollary 2.19. *[80, Corollary 2.2] If $S \in L(H)$ is paranormal then $J = S + G$ is finite, where G is a compact operator.*

Corollary 2.19 shows that a paranormal operator + compact operator is finite but the study was limited to establishment of orthogonality conditions for these finite operators.

Theorem 2.20. *[80, Theorem 2.4] If M is p-hyponormal(resp.log-hypornomal) and if V^* is $p - hypornormal(resp.log - hypornomal)$, then $\|Q - (MX - XM)\| \geq \|Q\|$ for all $X \in L(H)$ and $Q \in ker\delta_{M,V}$.*

Theorem 2.20 shows log-hyponormal operators are finite and it is shown that $Ran(\delta_{M,V})$ is orthogonal to $Ker(\delta_{M,V})$ but the study was limited to establishment of orthogonality conditions for these finite operators.

Bouzenda [13] proved that a spectraloid operator is finite and that the operator of the form $J+C$ is also finite given that C is a compact operator. Bouzenda [13] presented some results on generalized finite operators and gave a new set of finite operators. Then orthogonality results of some operators were given. The following are Bouzenda's main results:

Theorem 2.21. *[13, Theorem 1] Let $S \in L(H)$ be convexoid, then S is finite.*

Theorem 2.21 shows convexoid operators are finite but did not give a detailed description of finite operators.

Corollary 2.22. *[13, Corollary 1] The following operators are finite*

(i). Hyponormal operators,

(ii). Transaloid operators,

(iii). Paranormal operators,

(iv). Normaloid operators.

Corollary 2.22 shows hyponormal operators, transaloid operators, paranormal operators and nomaloid operators are finite but did not give a detailed description of finite operators.

Corollary 2.23. *[13, Corollary 2] Let $S \in Y$ be convexoid. Then $J = S + C$ is finite, where C is a compact operator.*

Corollary 2.23 shows that convexoid operator S plus a compact operator K is finite but did not give a detailed description of finite operators.

Theorem 2.24. *[13, Theorem 4] Let* $S \in L(H)$ *then for every normal operator* Q *such that* $SQ = QS$*, we have* $\|SX - XS - Q\| \geq \|Q\|$ *for all* $X \in L(H)$.

Theorem 2.24 considered normal operators and showed that the range of δ_S is orthogonal to its nullspace but did not characterize finiteness of these elementary operators.

Theorem 2.25. *[13, Theorem 2] Let* $S, P \in L(H)$*. If* $S, P \in Y^*$*, then* $\|SX - XS - Q\| \geq \|Q\|$ *for every* $X \in L(H)$ *and for every* $Q \in ker\delta_{S,P}$.

Theorem 2.25 shows that the range of $\delta_{S,P}$ is orthogonal to $Ker\delta_{S,P}$ but did not characterize finiteness of these elementary operators.

Hadia [48] presented some properties of finite operators and gave some group of operators which are in the class of finite operators and found for which condition $A + W$ is a finite operator in $L(H \oplus H)$. The following are Hadia's main results:

Proposition 2.26. *[48, Proposition 2.2] Let* $A, B \in L(H)$*, then* $AB \in F(H)$.

Proposition 2.26 shows some group of operators that are in the group of finite operators but did not characterize finiteness of elementary operators.

Lemma 2.27. *[48, Lemma 3.3] Suppose* $M \in L(H)$*, if* M *is a posinormal operator then* $M \in F(H)$.

Lemma 2.27 shows that posinormal operators are finite but did not give a detailed description of finite operators.

Lemma 2.28. *[48, Lemma 3.4] Suppose $J \in L(H)$, if J is a normaloid operator then $J \in F(H)$.*

Lemma 2.28 shows that nomaloid operators are finite but did not characterize finiteness of elementary operators.

Corollary 2.29. *[48, Corollary 3.8] Let $Q \in L(H)$, such that $A^n = 1$, for each $n \in \mathbb{N}$, then $\|QX - XQ - 1\| \geq \|1\|$ $\forall$ $X \in L(H)$ i.e $Q \in F(H)$.*

Corollary 2.29 shows orthogonality of elementary operators but did not characterize finiteness of these elementary operators.

In [78] a set of finite operators was presented by Salah which consists of the set of paranormal operators and proved the range-kernel orthogonality for paranormal operators. The following are Salah's main results:

Theorem 2.30. *[78, Theorem 2.5] Suppose $N \in L(H)$ is spectraloid, then N is finite.*

Theorem 2.30 shows that spectraloid operators are finite but did not characterize finiteness of elementary operators.

Lemma 2.31. *[78, Lemma 2.7] If J is normal operator and Q is class y such that $QJ = JQ$ then for every $\lambda \in \delta_p(J)$ we have $\|J - QX - XQ\| \geq \|J\|$ for all $X \in L(H)$*

Lemma 2.31 considered a class y operator and showed that the range of δ_Q is orthogonal to its null space but did not characterize finiteness of elementary operators.

Theorem 2.32. *[78, Theorem 2.8] Suppose Q is class y then for all normal operator J such that $QJ = JQ$ we have $\|J - QX - XQ\| \geq \|J\|$ $\forall\ X \in L(H)$.*

Theorem 2.32 considered a class y operator and showed that the range of δ_A is orthogonal to $Ker\delta_A$ but did not characterize finiteness of elementary operators.

Corollary 2.33. *[78, Corollary 2.10] If $S \in L(H)$ is a class y. Then $J = S + Q$ is finite, where Q is a completely continuous operator.*

Corollary 2.33 shows a spectraloid plus a completely continuous operator are finite but did not characterize finiteness of elementary operators.

Salah and Smail [84] proved that a paranormal operator is finite and presented properties of finite operators. The following are Salah's main results:

Theorem 2.34. *[84, Theorem 2.1] Suppose $T \in L(H)$ be paranormal, then T is finite.*

Theorem 2.34 shows that paranomal operators are finite but did not characterize finiteness of elementary operators.

2.3 Orthogonality in normed spaces

Kapoor and Jagadish Prasad [59] provided new characterization of inner product spaces and some proofs similar to the existing characterizations were given. However, it was proved that in a vector space in which a norm is defined both Pythagorean and Isosceles orthogonalities are unique. The following are some of their results:

Theorem 2.35. *[59, Theorem 3] (a) Isosceles orthogonality in Y is distinctive given that Y is strictly convex. (b) Pythagorean orthogonality is distinctive.*

Theorem 2.35 characterizes uniqueness of Isosceles and Pythagorean orthogonality but did not determine Birkhoff-James orthogonality for finite elementary operators.

Theorem 2.36. *[59, Theorem 4] For space Ω that is normed the following are equivalent:*

(i). Ω *is an inner product space,*

(ii). $q, g \in \Omega, q \perp_P g \Rightarrow g \perp_I q,$

(iii). $q, g \in \Omega, q \perp_I g \Rightarrow g \perp_P q.$

Theorem 2.36 shows that in a linear space that is defined by an inner product both Pythagorean and Isosceles orthogonalities have similar properties but did not determine Birkhoff orthogonality for finite elementary operators.

Theorem 2.37. *[59, Theorem 5] Given a normed linear space Ω the following are equivalent:*

(i). Ω *is an inner product space,*

(ii). $q, g \in \Omega, q \perp_P g \Rightarrow g \perp_J q,$

(iii). $q, g \in \Omega, q \perp_J g \Rightarrow g \perp_P q.$

Theorem 2.37 shows that in a vector space in which an inner product is defined Pythagorean orthogonality is equivalent to Birkhoff-James orthogonality but did not establish orthogonality conditions for finite elementary operators.

Theorem 2.38. *[59, Theorem 5] For a space* Ω *that is normed the following are equivalent:*

(i). Ω *is an inner product space*

(ii). $j, q \in \Omega, j \perp_J q \Rightarrow q \perp_I j.$

(iii). $j, q \in \Omega, j \perp_I q \Rightarrow q \perp_J j.$

Theorem 2.38 shows that in inner product spaces Isosceles orthogonality and Birkhoff-James orthogonality are equivalent but did not establish orthogonality conditions for finite elementary operators.

Diminnie, Raymond and Edward [26] defined another type orthogonality in normed spaces that involves pythagorean orthogonality and Isosceles orthogonality and it was shown that a new orthogonality is homogenous in an inner product space. The following are their main results:

Theorem 2.39. *[26, Theorem 1] If* α*-orthogonality is homogenous or additive, then the space* $(Y, \|.\|)$ *is a real inner product space.*

Theorem 2.39 establishes the homogeneity condition for α-orthogonality and it was shown that α-orthogonality is homogenous for a inner product space but did not establish the finiteness condition for elementary operators in normed spaces .

Theorem 2.40. *[26, Theorem 1.2] If α-orthogonality is homogenous, then $(g \perp h)(a)$ if $\|i-l\|^2 = \|i\|^2 + \|l\|^2$ i.e α-orthogonality is equivalent to pythagorean orthogonality.*

Theorem 2.40 establishes the homogeneity condition for α-orthogonality and it was shown that α-orthogonality is homogenous if and only if it is equivalent to Pythagorean orthogonality but did not establish the finiteness of elementary operators in normed spaces .

Koldobsky [56] showed that a linear operator $J : Y \to Y$ preserves orthogonality given that J is isometric and is multiplied by a positive constant. The following are some of his main result:

Lemma 2.41. *[56, Lemma 1] Suppose $\alpha \in D(y, z), a, b \in \mathbb{R}$. Then*

(i). $y^*(ay + bz)$ does not rely $y^* \in S(y + az)$.

(ii). $y+\alpha z \perp ay+bz \ \ if \ \ if \ \ y^*(ay+bz) = 0 \ \ for \ \ every \ \ y^* \in S(y+az)$.

Lemma 2.41 shows that $y + \alpha z \perp ay + bz \ \ if \ \ and \ \ only \ \ if \ \ y^*(ay + bz) = 0$ but did not determine orthogonality conditions for finite elementary operators.

Lemma 2.42. *[56, Lemma 2] Let α be a group of numbers such that $w + az \perp z$ is a closed segment $[m, M]$ in $\mathbb{R}$ and $\|w + \alpha z\| = \|w + mz\|$.*

Lemma 2.42 shows convexity of the function $\alpha \to \|w + \alpha z\|$ but did not determine Birkhoff orthogonality for finite elementary operators.

Theorem 2.43. *[56, Theorem 3] Let Y be a Banach space and $J : Y \to Y$ be a linear operator preserving orthogonality. Then $J = kV \ \ where \ \ k \in \mathbb{R} \ \ and \ \ V$ is an isometry.*

Theorem 2.43 shows that a linear map $J : Y \to Y$ preserves orthogonality provided that J is an isometry but did not determine Birkhoff orthogonality for finite elementary operators.

Alonso and Maria [4] studied geometric properties of an orthogonality relation based on a classical property of right angles defined in normed linear spaces but did not determine Birkhoff orthogonality for finite elementary operators.

Jacek [53] defined an approximate Birkhofff orthogonality relation in normed spaces. Jacek [53] related it with that given by Dragomir and established some properties .It was shown that in smooth space approximate Birkhofff orthogonality is equal to the approximate orthogonality from the semi-vector in which an inner product is defined. The following are Jacek's main results:

Proposition 2.44. *[53, Proposition 2.2] If Y is an inner product space then for arbitrary $\varepsilon \in [0, 1]$, $w \perp^{\varepsilon} z \Leftrightarrow w \perp_B^{\varepsilon} z$.*

Proposition 2.44 shows that w is orthogonal to z in a space endowed with an inner product implies that w is Birkhoff-James orthogonal to z but the study was limited to establishment of orthogonality conditions for finite elementary operators.

In [28] Dragoljub introduced ψ Gateaux derivative for operators to be orthogonal(in the sense of James) to the operator in both spaces C_1 and C_∞ (nuclear and compact operators on a Hilbert space). Further Dragoljub [28] applied these results to prove that there exists a normal derivation δ_A such that $\overline{ran\delta_A} \oplus ker\delta_A \neq C_1$ and a related result concerning C_∞. The following are some of their main results:

Theorem 2.45. *[28, Theorem 1.4] The vector z is orthogonal to w in terms of James if and only if $inf\psi D_{\psi,x}(z) \geq 0$.*

Theorem 2.45 characterizes orthogonality in Banach spaces (without care of smoothness) via ψ-Gateaux derivative but did not determine Birkhoff-James orthogonality for finite elementary operator.

Fathi [40] adopted the notion of orthogonality and established a characterization for orthogonality in the spaces $L_q^p(K)$, $\infty > P \geq 1$, given that Q is a group of integers that are non-negative. Finally, orthogonality in the Hilbert spaces $L_s^2(K)$ was characterized through inner products but did not establish orthogonality conditions for finite elementary operators.

Hendra and Mashadi [49] discussed some types of orthogonalities in 2-normed spaces and their drawbacks. Hendra and Mashadi [49] also formulated new definitions of orthogonality that improved the existing ones. In the standard 2-normed spaces their types of orthogonality are similar with the usual one. The following are some of the main results obtained:

Proposition 2.46. *[49, Theorem 3.3] Suppose $(Y, \|.\|)$ is the standard 2-normed space of dimension 3 or higher. If $v \perp_G w$, then $v \perp w$.*

Proposition 2.46 shows that w is orthogonal to v if w is general orthogonal to v but did not establish orthogonality conditions for finite elementary operators.

Madjid and Mohammad [69] introduced the notion of orthogonality constant mappings in isosceles orthogonal spaces and established stability of orthogonal constant mappings then finally the stability of periderized quadratic equation $f(w+v)+g(w+v)=h(w)+h(v)$ was studied. Madjid and Mohammad [69] dealt with isosceles orthogonality and in their case isosceles orthogonal space was a normed space Y with the isosceles orthogonality. The following are some of their main results:

Lemma 2.47. *[69, Lemma 2.2] Suppose $K: Y \to Z$ is an orthogonality constant mapping. If $w, z \in Y$ and $\|w\| = \|z\|$ then $k(w) = k(z)$.*

Lemma 2.47 shows constant mappings in isosceles orthogonal spaces but did not determine Birkhoff-James orthogonality for finite elementary operators.

Shoja and Mazaheri [85] investigated properties of the general orthogonality in Banach spaces and obtained some results of general orthogonality in Banach spaces are the same as those of orthogonality in Hilbert spaces. The relation between this concept in smooth spaces and sense of Birkhoff-James was also considered. The following are some of the main results obtained:

Theorem 2.48. *[85, Theorem 2.2] If $\mathbb{R}$ is a normed space. Then the following properties are true.*

(i). If $k, r \in \mathbb{R}$, $k \perp^{G} r$, then $r \perp^{BG} k$.

(ii). If $k = 0 \in \mathbb{R}$ is a normal element, $r \in \mathbb{R}$ and $k \perp^{BG} r$, then $k \perp^{G} r$.

Theorem 2.48 shows that j is Birkhoff-James orthogonal to r if k is general orthogonal to r but did not establish orthogonality conditions for finite elementary operators.

Theorem 2.49. *[85, Theorem 2.7] If D is a real Banach space, $x \in D$ and $Q \subseteq D$. Let the General orthogonality be G-additivity. Then there exist a unique $y_o \in Q$ such that $x - y_0 \perp^G y$.*

Theorem 2.49 shows some properties of the General orthogonality in Banach spaces but did not establish orthogonality conditions for finite elementary operators.

Jacek [52] considered Birkhoff-James orthogonality in a space in which a norm is defined and a class of linear mappings that approximately preserve this relation. Some related stability problems were stated. The following are Jacek's main results:

Theorem 2.50. *[52, Theorem 2.9] Let Z be a normed space. If there exist an inner product space S and a transformation r from Z into S or from S onto Z such that r is Birkhoff-James orthogonal. Then Z is an inner product space.*

Theorem 2.50 shows Z is in a vector space with an inner product if r preserves the Birkhoff-James orthogonality but did not determine Birkhoff-James orthogonality for finite elementary operators.

Horst Martin [46] showed that $Q : W \to Z$ preserves Isosceles orthogonality provided it is an isometric scalar multiple. The following are his main results:

Lemma 2.51. *[46, Lemma 4] Suppose W and Z are normed linear spaces. Given a linear transformation $Q : W \to Z$ is Isosceles orthogonal, then it also Birkhoff orthogonal.*

Lemma 2.51 establishes condition for an operator to preserve Isosceles orthogonality, it is shown that if an operator preserves Isosceles orthogonality, then it preserves Birkhoff orthogonality but did not establish orthogonality conditions for finite elementary operators.

Theorem 2.52. *[46, Theorem 5] Suppose Q and Z are normed linear spaces. A linear map $R : Q \to Z$ is Isosceles orthogonal given by R is a linear isometric scalar multiple.*

Theorem 2.52 establishes condition for an operator to preserve Isosceles orthogonality and it is shown an operator preserves Isosceles orthogonality given that R is a linear isometric scalar multiple but this study was limited to establishment of orthogonality conditions for finite elementary operators.

Kallol and Hossein [61] obtained properties for a linear transformation T to be orthogonal to other linear transformation A in terms of James. Also it was shown that if T is orthogonal to A and $O \notin \delta_p(A)$ then $sup\{|(Tu, v)| = \|u\| = 1$ and $(Au, v) = 0\}$. It was proved that the complex scalar λ_0 is characterized by the fact that there exist $\{x_n\}, \|x_n\| = 1$ such that $((T - \lambda_o A)_{x_n}, Ax_n) \to 0$ and $\|(T - \lambda_0 A)_{x_n}\| \to \|T - \lambda_0 A\|$.

Theorem 2.53. *[61, Theorem 1] Suppose Q and G are two linear transformations on a complex Hilbert space Z. Then there exist a complex scalar λ_0 such that $\|Q - \lambda_0 G\| = \|Q - \lambda G\|$.*

Theorem 2.53 shows that if T is orthogonal to A and $O \notin \delta_p(A)$ but did not determine Birkhoff-James orthogonality for finite elementary operators.

Dragomir and Kikianty [29] introduced types of orthogonality in form of 2-HH norms and then a further study on 2-HH norms was done. Inner product spaces were characterized together with with strictly convexity of spaces. The following are some of their main results:

Theorem 2.54. *[29, Theorem 3.5] Suppose W is a normed linear space. Then HH-P orthogonality is homogenous provided W is an inner product space*

Theorem 2.54 shows that HH-P orthogonality is homogenous given that W is an inner product space but did not determine Birkhoff-James orthogonality for finite elementary operators.

Theorem 2.55. *[29, Theorem 3.6] If W is a normed linear space. Then HH-I orthogonality is homogenous provided W is an inner product space*

Theorem 2.55 shows that HH-I orthogonality is homogenous given that W is an inner product space but did not determine Birkhoff-James orthogonality for finite elementary operators.

Lemma 2.56. *[29, Lemma 3.1] The additivity and homogeneity of HH-P and HH-I orthogonality are equil.*

Lemma 2.56 shows that HH-I and HH-I orthogonality are equivalent but did not determine Birkhoff-James orthogonality for finite elementary operators.

Khalil and Alkhawaida [64] presented two new definition of orthogonality types. One is related to proximity in Banach spaces and other related to contractive projections. The relation between the two types was studied and basic properties of each type were presented. The reflection of such orthogonalities to compact operators was discussed. The following are their main results:

Theorem 2.57. *[64, Theorem 3.2] Suppose Z is a Banach space. Then $j \perp^d h$ if $j \perp^n h$.*

Theorem 2.57 shows that j is d-orthogonal to h if $j \perp^n h$ but did not establish orthogonality conditions for finite elementary operators.

Theorem 2.58. *[64, Theorem 3.4] Let Z be a Banach space with factorization. If the d-orthogonality is additive, then Z is a Hilbert space.*

Theorem 2.58 presented additive property of d-orthogonality but did not establish orthogonality conditions for finite elementary operators.

Hossein [47] extended the usual concept of orthogonality to Banach spaces. Completely continuous operators were characterized on Banach spaces that posses an orthonormal countable basis was also established. The following are Hossein's main results:

Theorem 2.59. *[47, Theorem 2] Suppose $(t_k)_k \in Z$ be a sequence in E, the following are equivalent:*

(i). The sequence $(t_k) \in Z$ is orthogonal in E.

(ii). For every pair of sequences $(g_k)_k \in Z$ and $(h_k)_k \in K$ satisfying $|g_k| = |h_k|$ for every $k \in Z$, $\sum_{k \in Z} h_k x_k$ converges if $\sum_{k \in Z} h_k x_k$ converges and if both $\| \sum_{k \in Z} h_k x_k \| = \| \sum_{k \in Z} h_k x_k \|$.

Theorem 2.59 characterized completely continuous operators on Banach spaces that admit orthonormal countable basis but the study was limited to characterization of finite elementary operators.

Cuixia and Senlin [24] studied isosceles orthogonality is homogenous, and that was the crucial orthogonality type in normed linear spaces, from the given view point. To add, Cuixia and Senlin [24] studied the link between homogeneity of isosceles orthogonality and other concepts such as isometric reflection vectors and l_2-summand vectors, It was proved that a Banach space Z is a Hilbert space given that the interior of the class of homogeneity of isosceles orthogonality in the unit sphere of Q is non-empty. In addition, a geometric constant NH_Q was introduced to show that the isosceles orthogonality is not homogenous . It was shown that $0 \leq NH \leq 2$ $NH_X = 0$ given that Z is a Hilbert space and $NH_Q = 2$ provided that Z is non-uniformly square.

Lemma 2.60. *[24, Lemma 2] Suppose $z \in H_Z$ is a point of S_Z that is smooth then z is an isometric reflection vector, and hence z is Robert orthogonal to a hyperplane.*

Theorem 2.60 shows that z is Rorberts orthogonal if z is an isometric reflection vector but did not determine Birkhoff-James orthogonality for finite elementary operators.

Salah and Hacene [82 minimized the C_∞-norm of affine maps from $L(Z)$ to C_∞ by use of convex and differentiable analysis as was an studied in operator theory. The transformations considered generally the elementary operator especially the generalized derivations that were most important.

As a consequence, global minima was characterized in terms of orthogonality. The following are some of their main results:

Theorem 2.61. *[82, Theorem 3.3] Suppose $Q \in C_\infty$, $\varphi(Q)$ has the polar decomposition $\varphi(Q) = U|\varphi(Q)|$ and let $j \in T$. Then $|Q + (SX - XR)|_{C_\infty} \geq \|\varphi(Q)\|_{C_\infty}$; $(j \otimes Uj) \in Ker_{R^*S}$ for all $X \in C_\infty$.*

Theorem 2.61 characterizes the orthogonality of operators Q in C_∞ but did not determine Birkhoff-James orthogonality for finite elementary operators.

Corollary 2.62. *[82, Corollary 3.2] Suppose $Q \in C_\infty \bigcap Ker\delta_{S,R}$, $\varphi(Q)$ has the decomposition $\varphi(S) = U|\varphi(Q)'|$ and let $j \in \Gamma$. Then the following assertions are equivalent.*

(i). $|Q + (SX - XR)|_{C_\infty} \geq \|\varphi(Q)\|_{C_\infty}$ *for all* $X \in C_\infty$,

(ii). $(j \otimes Uj) \in Ker_{R^*S^*}$.

Corollary 2.62 characterizes the orthogonality of operators Q in C_∞ but did not determine Birkhoff-James orthogonality for finite elementary operators.

Theorem 2.63. *[82, Theorem 3.4] Suppose $Q, P \in C_\infty$ and $a \in T$, where $Q = H|Q|$ is a point in C_∞. The following properties are saidto be equivalent.*

(i). F_ψ *on* C_∞ *has a global at* Q.

(ii). *max* $j \in \lambda$, $\|j\| = 1$ $Re\langle \phi(P), j, Uj\rangle \geq 0$.

(iii). for every $P \in C_\infty$ $tr((j \otimes' Uj)\phi(P)) = 0$.

(iv). $\phi(P)j \perp Qj$.

Theorem 2.63 characterizes orthogonality in terms of Birkhoff but did not determine Birkhoff-James orthogonality for finite elementary operators.

Later Ionica [50] presented some relationships between Birkhoff orthogonality and some concepts in convex analysis. Through this Ionica [50] obtained Blanco and Turnsek's results regarding linear mappings which has Birkhoff orthogonality. The following are his main results:

Theorem 2.64. *[50, Theorem 1] If* $P : X_1 \to Y_2$ *is a linear transformation, then the properties follow are the same:*

(i). P *has the orthogonality.*

(ii). suppose $\acute{n}_{1+}\langle q, r\rangle = 0, q, r \in X_1$, *then* $\acute{n}_{2+}\langle Pq, Pr\rangle \geq 0$.

(iii). $F_{q,r}$ *is not reducing on* $\mathbb{R}$.

(iv). $F_{q,r}$ *is constant on* $\mathbb{R}$.

(v). There is $g > 0$ *such that* $\|Pq\|_2 = g\|q\|_1$, *for* $q \in X_1$.

Theorem 2.64 establishes properties that characterize the linear operators which preserve orthogonality but did not characterize finiteness of operators.

Corollary 2.65. *[50, Corollary 1] Orthogonalities on the similar normed space* W *are equal given that the corresponding norms are proportional.*

Corollary 2.65 establishes a simple property which characterizes the linear operators preserving orthogonality but did not characterize finiteness of elementary operators.

Theorem 2.66. *[50,Theorem 1] The statements that follow are equivalent:*

(i). A linear operator $Q : X_1 \to Y_2$, where X_1, Y_2 are two normed spaces, is orthogonal given that Qis an isometric.

(ii). A linear operator $Q : X_1 \to Y_2$, where X is a normed space, is orthogonal given that Q an isometry multiplied by a constant that i positive.

(iii). Given orthogonality types on the similar normed space X, characterized by two norms on X, are equal provided the two norms are proportional.

Theorem 2.66 establishes properties that characterize the orthogonality of linear operators but did not characterize finiteness of elementary operators.

Ali Zamani and Mohammad [9] investigated properties of approximate Roberts orthogonality and how they are related to approximate Birkhoff orthogonality. Also they studied the set of linear mappings that are approximate Roberts orthogonal of type $\varepsilon \perp R$. It was shown that an ε-isometric scalar multiple preserves approximate Roberts orthogonality. The following are some of their main results:

Proposition 2.67. *[9, Proposition 2.1] Suppose $\varepsilon \in [0,1)$. Then $\perp_R^{\varepsilon}$ and $^{\varepsilon}\perp_R$ are symmetric i.e* *(i).* $j \perp_R^{\varepsilon} k \Rightarrow k \perp_R^{\varepsilon} j$, $\forall j, k \in Z$

(ii). $j^{\varepsilon} \perp_R k \Rightarrow k^{\varepsilon} \perp_R j$, $\forall k, j \in Z$

Proposition 2.67 shows that approximate Roberts orthogonality is symmetric but did not determine Birkhoff-orthogonality for finite elementary operators.

Proposition 2.68. *[9, Proposition 2.2] Let $\varepsilon \in [0, 1)$. Then $\perp_R^{\varepsilon}$ and $^{\varepsilon} \perp_R$ are homogeneous i.e*

(i). $j \perp_R^{\varepsilon} k \Rightarrow \alpha j \perp_R^{\varepsilon} \beta k$, $\forall j, k \in Z$

(ii). $j^{\varepsilon} \perp_R k \Rightarrow \alpha j^{\varepsilon} \perp_R \beta k$, $\forall j, k \in Z$

Proposition 2.68 shows that approximate Roberts orthogonality is homogenous but did not determine Birkhoff-orthogonality for finite elementary operators.

Proposition 2.69. *[9, Proposition 2.3] If Z is a normed space that is and $q, r \in Z$ then $q \perp_R^{\varepsilon} r \Rightarrow q^{\varepsilon} \perp_R r$ for any $\varepsilon \in [0, 1)$.*

Proposition 2.69 shows that approximate Roberts orthogonality is symmetric but did not determine Birkhoff-orthogonality for finite elementary operators.

Justyna [54] showed how different concepts of orthogonality have been discussed in functional equations. Orthogonality relations were introduced and functional equations examples were given intended for only orthogonal vectors. Some of solutions of functional equations together with some applications were shown. Then the problem of stability taking into account different aspects of the problem was discussed. Also Justyna [54] mentioned the orthogonality equation and the problem preserving orthogonality. Moreover, after proofing results, some open problems regarding those topics were stated but did not determine Birkhoff-orthogonality for finite elementary operators.

Pawel [74] introduced an approximate orthogonality relation and considered groups of linear mappings with approximate orthogonality. Pawel [74] showed that especially the property that a transformation that preserve B-orthogonality is equal to the orthogonality that preserves the p, p_+-orthogonality, although the mentioned orthogonalities do not necessarily need to be equivalent. In addition, it was shown that every linear mapping with approximate orthogonality is mainly a isometric scalar multiple. It was shown that a linear map with Birkhoff-James orthogonality is an isometric scalar multiple. Pawel [74] gave some characterizations of linear mappings with approximate orthogonality in real normed spaces but in this study we have determined Birkhoff-orthogonality for finite elementary operators.

Later in [75] Pawel extended this study and showed that the semi-orthogonality type and p_+-orthogonality of approximate are generally not comparable unless and otherwise in a smooth normed space. Consequently, smooth spaces were determined in relation to approximate orthogonality was given but did not determine Birkhoff-orthogonality for finite elementary operators.

In [10] Ali Zamani and Mohammad introduced the notion of the set of approximate Roberts orthogonality and investigated the conditions of those sets geometrically. However, Ali Zamani and Mohammad [10] introduced a-isosceles orthogonality concept that is approximate and considered a group of mappings, that are approximately a-isosceles orthogonal but did not determine Birkhoff-orthogonality for finite elementary operators.

Chaoqun and Fangyan [25] investigated maps that are norm derivative orthogonal between normed spaces. Those maps were shown to be a scalar multiple of an isometry but did not determine Birkhoff-orthogonality for finite elementary operators.

Bhuwan [20] studied two new types of orthogonality from generalized Carlsson orthogonality and some properties of orthogonality in Banach spaces were verified as Best implied Birkhoff and Birkhoff type of orthogonality implied Best approximation. It was also shown that Pythagorean implies Best approximation. The following are some of Bhuwan's main results:

Theorem 2.70. *[20, Theorem 2.2] If Z is a a real normed space and G is a subspace of Z. Then, $y_0 \in PG(X)$ given that $(x - y_0) \perp BG$.*

Theorem 2.70 shows that Birkhoff orthogonality, Pythagorean orthogonality and Isosceles orthogonality, implies best approximation but did not determine Birkhoff-orthogonality for finite elementary operators.

Lemma 2.71. *[20, Theorem 2.3] Suppose Z is a normed space, if for $x \in Z$ there exist $y_0 \in G$ and that $x - y_0 \in P(G)$ then $y_0 \in PG(x)$.*

Lemma 2.71 shows that Rorbert orthogonality, Pythagorean orthogonality and Isosceles orthogonality implies best approximation but did not determine Birkhoff-orthogonality for finite elementary operators.

In [33] Debmalya, Kallol and Arpita studied the concepts Birkhoff-James orthogonality that is approximate in a vector space in which a norm is defined, geometrically, they were characterized in form of normal cones. Interconnection betweeen normal cones and approximate Birkhoff-James orthogonality to characterize normal cones completely in a Banach space of dimension two was explored. Theorem for uniqueness of approximate Birkhoff-James orthogonality set in a vector space in which a norm is defined was also obtained. Their main aim was to study two different types of approximate Birkhoff-James orthogonality, to be able to understand the geometry of normed spaces. Among other things Debmalya, Kallol and Arpita [33] exhibited that the two aspects of approximate Birkhoff-James orthogonality have a close connection with normal cones in a in a vector space in which a norm is defined but did not determine Birkhoff-orthogonality for finite elementary operators.

Thomas [91] combined functional analytic and geometric view points on approximate Birkhoff orthogonality in generalized minkowskis spaces which are finite dimensional vector spaces equipped with a gauge. That was the first approach in those spaces but in our study we determined Birkhoff-orthogonality for finite elementary operators.

Ghosh, Debmalya and Kallol [44] explored the strict convexity of a vector space Z in which a norm is defined and orthogonality of operators through Birkhoff-James in $K(Z)$, the space of all completely continuous operators on Z. It was proved that a reflexive Banach space Z is said to be convex if given $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$ or $Rz = 0$ for some $z \in S_z$

with $\|Qz\| = \|Q\|$. It was shown that if Z is a Hilbert space of infinite dimension then for all $Q \in L(Z)$ $R \perp_B Q \Rightarrow Q \perp_B R$ given that Q is the operator of zero. It was also shown that $R \perp_B Q \Rightarrow Q \perp_B R$ for a real Hilbert space Z, $Q \perp_B R \Rightarrow R \perp_B Q$ for $R \in L(Z)$ provided Q is the zero operator. The following are some of their main results:

Theorem 2.72. *[44, Theorem 2.1] Suppose X is a Banach space that is strictly convex and reflexive. Then for $Q, R \in K(X)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$ or $Rz = 0$ for some $z \in M_Q$.*

Theorem 2.72 shows that a Banach space X is reflexive and convex if for any $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$ but in our study we determined Birkhoff-orthogonality for finite elementary operators.

Theorem 2.73. *[44, Theorem 2.2] If Z is a a Banach space that is strictly convex and $R \in K(Z)$ is an injective. Then given $Q \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$.*

Theorem 2.73 shows for $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$ but did not determine Birkhoff-orthogonality for finite elementary operators.

Theorem 2.74. *[44, Theorem 2.4] Suppose Z is a real normed linear space and if $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$ or $Rz = 0$ for some $z \in M_Q$. Then Z is strictly convex.*

Theorem 2.74 shows that a real normed linear space Z is strictly convex provided for $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$ but did not determine Birkhoff-orthogonality for finite elementary operators.

Theorem 2.75. *[44, Theorem 2.5] Suppose Z is a real reflexive Banach space. Then Z is strictly convex given that for any $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$ or $Rz = 0$ for some $z \in M_Q$.*

Theorem 2.75 shows that a real reflexive Banach space Z is strictly convex given $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$ but did not determine Birkhoff-orthogonality for finite elementary operators.

Corollary 2.76. *[44, Corollary 2.6] If Z is a real finite dimensional normed space. Then Z is strictly convex if for $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$ or $Rz = 0$ for some $z \in M_Z$.*

Corollary 2.76 shows that a normed space Z of finite dimension satisfies the strictric convexity property given that $Q, R \in K(Z)$, $Q \perp_B R \Rightarrow Q \perp_{SB} R$ but did not determine Birkhoff-orthogonality for finite elementary operators.

Debmalya [30] characterized Birkhoff-James orthogonality of linear tansformations on a real Banach space of finite dimension. Debmalya [30] then explored the symmetric pproperties of Birkhoff-James orthogonality of operators that are bounded defined on Z. It was proved that $Q \in L(l_p^2)$ $(p \geq 2, p \neq \infty)$ has left symmetry in relation to Birkhoff-James orthogonality given that Q is the zero operator. Debmalya [30] concluded that the result holds for any real Banach space of finite dimension that is strictly convex l_p^n $(p > 2, p \neq \infty)$. The following are some of Debmalya's main results:

Theorem 2.77. *[30, Theorem 2.2] Let Z is a real Banach space of finite dimension. If $Q, R \in L(Z)$, then $Q \perp_B Z$ if there is $z, w \in M_Q$ such that $R_z \in Q_z^+$ and $R_w \in Q_z^-$.*

Theorem 2.77 shows how Birkhoff-James orthogonality have been characterized for linear operators but did not determine Birkhoff-orthogonality for finite elementary operators.

Corollary 2.78. *[30, Corollary 2.2.1] Suppose Z a real Banach space of finite dimension. If $Q \in L(Z)$ is such that Q attains norm only at P, where P is a subset of S_z that is connected and not open. Then for $R \in L(Z)$ with $Q \perp_B R$, there is $x \in P$ where $Q_z \perp R_z$.*

Corollary 2.78 shows that $Q_z \perp R_z$ on a real Banach space of finite dimension but did not determine Birkhoff-orthogonality for finite elementary operators.

Jacek [51] considered a linear operator $Q : Z \to Z$ on a normed space Z reversing orthogonality. That is, having the condition $j \perp k \Rightarrow Q_k \perp Q_j$ $\forall j, k \in Z$ where $\perp$ stands for Birkhoff orthogonality. The following are some of Jacek's main results:

Theorem 2.79. *[51, Theorem 3.1] Suppose Z a Minkowski plane that satisfies either: (i) smoothness and not strict convexity or: (ii) strict convexity but not smoothness. Then there are no nontrivial linear operators with reverse orthogonality.*

Theorem 2.79 shows that in a Minkowski plane there are linear transformations that are not trivial with reverse orthogonality but did not determine Birkhoff-orthogonality for finite elementary operators.

Corollary 2.80. *[51, Corollary 3.2] If Z a Minkowski plane with a linear operator that is not zero with reverse orthogonality. Then Z is either smooth and has strict convexity or none holds.*

Corollary 2.80 shows the smoothness and strict convexity of Z given that Z is a Minkowski plane then Z is strictly convex but did not determine Birkhoff-orthogonality for finite elementary operators.

Kallol and Debmalya [60] studied Birkhoff-James orthogonality of operators Q, P on $(\mathbb{R}^n, \|.\|_\infty)$ and found a property for Q to be orthogonal to P in terms of Birkhoff-James with some properties on Q. In [60], a property for the existence of two operators Q, P on $(\mathbb{R}^n, \|.\|_\infty)$ with $Q \perp_B P$ such that $x \notin \mathbb{R}^n$ with $\|z\|_\infty = 1$, $Q_z \perp_B P_z$ and $\|Q_z\|_\infty = \|Q\|_\infty$ was found. Kallol and Debmalya [60] found required condition on Q so that if $Q_z \perp_B P_z$ then there exist $z \in \mathbb{R}^n$ with $\|z\|_\infty = 1$ such that $Q_z \perp_B P_z$ and $\|Q_x\|_\infty = \|Q\|_\infty$. The relationship between the orthogonality of vectors in $(\mathbb{R}^n, \|.\|_\infty)$ and the orthogonality of operators on $(\mathbb{R}^n, \|.\|_\infty)$ was also obtained. The following are some of Kallol and Debmalya's main results:

Theorem 2.81. *[60, Theorem 2.1] Let* $Q = (a_{ij})_{n\times n}$ *and* $P = (b_{ij})_{n\times n}$ *are two linear operators on* $(\mathbb{R}^n, \|.\|_\infty)$ *and there exist* $i_o \in \{1, 2, ...n\}$ *such that* $a_{ioj} \neq 0$ *for every* $j \in \{1, 2, ...n\}$ *and* $|a_{io1}| + |a_{io2}| + ... + |a_{ion}| > |a_{i1}| + |a_{i2}| + ... + |a_{in}|$ *for every* $i \in \{1, 2, ...n\}$ - i_o. *Then*

$\|Q\|_\infty \leq \|Q + \lambda P\|_\infty$ *for all* $\lambda \in \mathbb{R}$ *iff*
$(sgna_{io1})b_{io1} + ... + (sgna_{ion})b_{ion} = 0$ *where*

$$\begin{aligned} (sgna_{ij}) &= +1 \;\; if \;\; > 0 \\ &= -1 \;\; if \;\; < 0 \\ &= 0 \;\; if \;\; = 0. \end{aligned}$$

Theorem 2.81 gives a condition needed for the operators Q and P on $(\mathbb{R}^n, \|.\|_\infty)$ to be orthogonal in terms of Birkhoff-James but the study was limited to establishment of orthogonality conditions for finite elementary operators.

In [42] Ghosh, Kallol and Debmalya studied the orthogonality of Birkhoff-James for linear operators on $(\mathbb{R}^n, \|.\|_1)$. It was proved that $Q \perp_B P \Rightarrow P \perp_B Q$ for all operators Q on $(\mathbb{R}^n, \|.\|_1)$ provided Q obtains a norm at the point of extreme, image that has left symmetry for $(\mathbb{R}^n, \|.\|_1)$ and the other extreme points have zero images. In [42] it was also proved that $R \perp_B Q \Rightarrow Q \perp_B R$ for all operators Q provided Q obtains norm at all extreme points and images of extreme points are scalar multiples of extreme points. The following are their main results:

Lemma 2.82. *[42, Lemma 2.1] Multiples of scalar of a non-open unit ball are the point of* $(\mathbb{R}^n)$ *that has right symmetry.*

Lemma 2.82 shows that operators on $(\mathbb{R}^n, \|.\|_1)$ are symmetric in terms of Birkhoff-James but did not characterize finiteness of elementary operators.

Theorem 2.83. *[42, Theorem 2.2] Let* $T = (t_{ij})$ *be a linear operator on* $\mathbb{R}^n$*, then for any linear operator* A *on* $(\mathbb{R}^n)$ $A \perp_B T \Rightarrow T \perp_B A$ *provided* T *obtains norm at the points of extreme and images of the points of extreme are multiples of scalars.*

Theorem 2.83 establishes symmetric property for a linear operator in terms of Birkhoff-James that is $H \perp_B G \Rightarrow G \perp_B H$ given that G obtains norm at the points of extreme and images of the points of extreme are multiples of scalars $(\mathbb{R}^n)$ and images of the points of extreme are zero but did not determine Birkhoff-James orthogonality for finite elementary operators.

Theorem 2.84. *[42, Theorem 2.3] Let $F = (f_{ij})$ be a linear operator on $\mathbb{R}^n$, then for a linear operator J on $(\mathbb{R}^n)$ $J \perp_B F \Rightarrow F \perp_B J$ given that F obtains norm at only one extreme point and image of which is left symmetric point of $(\mathbb{R}^n)$ and images of other extreme points are zero.*

Theorem 2.84 establishes symmetric property for linear operators in terms of Birkhoff-James that is $J \perp_B F \Rightarrow F \perp_B J$ given that F obtains norm at all extreme points and image of that is left symmetric point of $(\mathbb{R}^n)$ and images of other extreme points are zero but did not determine Birkhoff-James orthogonality for finite elementary operators.

In [43] Ghosh, Debmalya and Kallol Studied orthogonality of linear operators in terms of Birkhoff-James defined on $(\mathbb{R}^n, \|.\|_\infty)$ and characterized the right and left symmetric operators on $(\mathbb{R}^n, \|.\|_\infty)$. The following are their main results:

Theorem 2.85. *[43, Theorem 2.1] Suppose $T = (t_{ij})$ is a nonzero operator on $\mathbb{R}^n$. For any linear operator A on $(\mathbb{R}^n)$ $A \perp_B T \Rightarrow T \perp_B A$ given that for each $i \in 1, 2, ...n$, exactly one term $t_{i2}, t_{i2}, ..., t_{in}$ is nonzero and of the same magnitude.*

Theorem 2.85 shows the study of operators on $(\mathbb{R}^n)$ in terms of Birkhoff-James, the theorem characterizes nonzero right symmetric linear operator on $(\mathbb{R}^n)$ but did not give orthogonality conditions for finite elementary operators.

Theorem 2.86. *[43, Theorem 2.3] Let Q be a linear operator on $\mathbb{R}^2$, then for the linear operator P on $(\mathbb{R}^2)$ $Q \perp_B P \Rightarrow P \perp_B Q$ provided Q obtains norm at the point of extreme , say e_i, Qe_i is a left symmetry point and the image of other extreme point is zero.*

Theorem 2.86 shows the study of operators on $(\mathbb{R}^n)$ in terms of Birkhoff-James, the theorem characterizes nonzero left symmetric linear operator on $(\mathbb{R}^2)$ but did not give orthogonality conditions for finite elementary operators.

Theorem 2.87. *[43, Theorem 2.5] Suppose Q is a linear operator on $\mathbb{R}^n, n \geq 3$. Then is a left symmetric given that Q is the zero operator.*

Theorem 2.87 establishes condition for an operator Q to be symmetric but did not establish orthogonality conditions for finite elementary operators.

In [57] Kallol, Debmalya and Arpita characterized the notion of approximate Birkhoff-James orthogonality in the group of operators that are bounded given on a vector space in which a norm is defined. Kallol, Debmalya and Arpita [57] characterized Birkhoff- James orthogonality in the algebra of transformations that are bounded given on Hilbert space was attained that led to the recent result by Chiemlink where Birkhoff-James orthogonality of transformations that are linear was characterized on Hilbert space of finite dimension and also operators that are bounded on Hilbert space of finite dimension and also completely continuous operators on any Hilbert space. The following their main results:

Theorem 2.88. *[57, Theorem 2.1] Suppose Z be a Banach space that is reflexive and W is a normed space. If $Q, R \in B(Z, W)$. Then for $0 \leq t < 1, T \perp_B^{\varepsilon} A$ if (i) or (ii) holds.*

(i). There exists $z \in M_Q$ *such that* $R_z \in (Q_z)^+$ *and for every* $\lambda \in (-1-\sqrt{1-\epsilon^2}-1+\sqrt{1-\epsilon^2})\frac{\|Q\|}{\|R\|}$ *there is* $z_\lambda \in S_Z$ *and that* $\|Q_{z\lambda}+\lambda R_{z\lambda}\| \geq 1-\sqrt{1-\epsilon^2}\|Q\|$.

(ii). There exists $w \in M_Q$ *such that* $R_w \in (Q_w)^-$ *and for every* $\lambda \in (-1-\sqrt{1-\epsilon^2}-1+\sqrt{1-\epsilon^2})\frac{\|Q\|}{\|R\|}$ *there is* $w_\lambda \in S_Z$ *and that* $\|Q_{w\lambda}+\lambda R_{w\lambda}\| \geq 1-\sqrt{1-\epsilon^2}\|Q\|$.

Theorem 2.88 characterizes approximate Birkhoff-James orthogonality on a reflexive Banach space but did not determine Birkhoff-James orthogonality for finite operators.

Theorem 2.89. *[57, 2.2] Suppose* Z *is a Banach space of finite dimension. If* $Q \in B(Z)$*. Then* $R \perp_B Q$ *if there exists* $w, z \in M_R$ *such that* $R_w \in (Q_w)^+$ *and* $R_z \in (Q_w)^-$*.*

Theorem 2.89 characterizes Birkhoff-James orthogonality of bounded linear operator in the norm attainment set but did not determine Birkhoff-James orthogonality for finite operators.

In [31] Debmalya, Kallol and Arpita studied Birkoff-James orthogonality on a vector space in that a norm is endowed that is not finite dimensional linear transformations that are bounded and characterization of Birkhoff James orthogonality of bounded linear transformations was also done. As a consequence, from the study, the authors [31] gave a determined Birkhoff-James orthogonality of bounded linear functionals on a real vector space in which a norm is defined given that the dual space is strictly convex. In [31] a necessary and required properties for smoothness of bounded linear transformations on a normed linear space of infinite dimension was provided. The following are some of their main results:

Theorem 2.90. *[31, Theorem 2.1] Suppose that Z is a reflexive Banach space and W be a normed linear space. Then for every $Q, R \in \mathbb{K}(Z, W)$ $Q \perp_B R$ given that x, $y \in M_Q$ such that $R_x \in (Q_x)^+$ and $R_y \in (Q_y)^-$.*

Theorem 2.90 gives the orthogonality Birkhoff-James for completely continuous operators on a Banach space but did not determine Birkhoff-James orthogonality for finite operators in normed spaces.

In [62] Kallol, Debmalya, Arpita and Kallidas studied Birkhoff-James orthogonality of linear transformations on complex complete vector space in which a norm is defined and obtained a complete characterization of the same. As a way of obtaining other definitions, it was illustrated that there is a possibility to determine orthogonality of completely continuous linear operators that are complex analogous to the real form. Hence, the theoretic property of Birkhoff-James orthogonality in the real vector space for operators in which a norm is defined could be given in form of corollaries to their resent study. As a fact, compact operators were characterized in the complex space in terms of Birkhoff-James orthogonality in order to differentiate the complex form from the real form. The left symmetric operators on complex two-dimensional l_p space given that T is the zero operator was also studied. The following are some of their main results:

Theorem 2.91. *[62, Theorem 2.4] Suppose Z is a Banach space with reflexivity and W is any normed linear space. Then for every $Q, R \in \mathbb{K}(W, Z)$. Then $Q \perp_B R$ if and only if for each $\alpha \in$*

U there is $x \in x(\alpha)$, $y \in y(\alpha) \in M_Q$ such that $R_x \in (Q_x)^+\alpha$ and $R_y \in (Q_y)^-\alpha$.

Theorem 2.91 shows how Birkhoff-James orthogonality of linear transformations is characterized on complex complete vector space in which a norm is defined but did not determine Birkhoff-James orthogonality for finite elementary operators.

Corollary 2.92. *[62, Corollary 2.10] Suppose Z is a strictly convex and smooth complex Banach space of finite dimension. If $Q \in L(Z)$ be such that their exists $w, z \in S_w$ satisfying*

(i). $w \in M_Q$

(ii). $z \perp_B w$

(iii). $Q_z \neq 0$. Then Q can not have symmetry.

Corollary 2.92 gives characterizes Birkhoff-James orthogonality of bounded linear transformations but did not determine Birkhoff-James orthogonality for finite elementary operators.

In [77] Sanati and Kardel described the algebra of operator that preserve orthogonality defined on a compete vector space Z of infinite dimension with an inner product as being a multiple scalar of operators that are unitary between Hilbert space Z and some subspaces of Hilbert space Z that are closed. It was shown that any circle (centred at the origin) is the spectrum of an orthogonality that preserves operator. The following are some of their main results:

Lemma 2.93. *[77, Lemma 2.1] Suppose Z is a Hilbert space and $Q \in L(Z)$. If Q preserves orthogonality, then so are Q^*Q and $\|Q\|$.*

Lemma 2.93 characterized the class of orthogonality preserving operators but did not determine Birkhoff-James orthogonality for finite elementary operators.

Corollary 2.94. *[77, Corollary 2.4] Let $Q \in OP(H)$ then $\|Qx\| = r(Q)\|x\| \; \forall x \in H$.*

Corollary 2.94 shows that the operator Q is orthogonality preserving then the norm of Qx is equivalent to the product of the spectral radius of Q and the norm of x but did not determine Birkhoff-James orthogonality for finite elementary operators.

Debmalya, Kallol, and Arpita [32] studied Birkhoff-James orthogonality of completely continuous operators between Hilbert spaces and Banach spaces. Using the concept of inner products that are semi in normed linear spaces and concepts that are linked geometrically, some of the recent results were generalized and improved. In particular, Euclidean spaces were characterized and it was proved that the norm of a completely continuous operator can be possibly obtained through of Birkhoff-James orthogonality set. Then best approximation type results were also presented in the space of bounded linear operators. The following are some of their main results:

Corollary 2.95. *[32, Corollary 2.2.1] Suppose Z is a Hilbert space of infinite dimension. If $Q, P \in K(Z, Z)$. Then $Q \perp_B P$ provided that $x \in M_Q$ such that $Q_x \perp P_x$.*

Corollary 2.95 shows orthogonality of operators on a Hilbert space through Birkhoff-James orthogonality but did not determine Birkhoff-James orthogonality for finite elementary operators.

In [68] Kallol presented results on Birkhoff-James orthogonality and smoothness in a vector space defined by a norm. Kallol [68] explored the orthogonality relation between elements in a complete vector space W in which a norm is defined and the space of linear mappings $L(W)$. Smoothness of the space of bounded linear operators was also studied. Kallol [68] obtained the following results:

Theorem 2.96. *[68, Theorem 2.2.1] Suppose $Q \in L(W, Z)$ and $M_Q = DU(-D)$ where D is not an empty connected subset of S_w. Then for any $R \in L(W, Z)$ $Q \perp_B R$ if there exists $w \in M_Q$ such that $Q_w \perp R_w$.*

Corollary 2.96 shows orthogonality relation between elements in a complete vector space W in which a norm is defined and the space of linear mappings $B(W)$ but did not establish orthogonality conditions for finite elementary operators.

In [21] Bhuwan and Prakash characterized orthogonality in a vector space in which a norm is defined in the best approximation. Therefore, it was discovered that Birkhoff-james orthogonality means best approximation and best approximation means Birkhoff-james orthogonality. Also it was shown that for ε-orthogonality, ε-best approximation means ε-orthogonality. In [21] Bhuwan and Prakash established the relation between pythagorean orthogonality and best approximation and also isosceles orthogonality and ε-best approximation in normed spaces but did not determine Birkhoff-orthogonality for finite elementary operators.

In [11] Ali Zamani considered a semi-inner product and generalized the notion of Birkhoff-James orthogonality of operators a vector space in which a complete inner product is defined. Moreover, the relation $Q \perp_A^B R$ was introduced given that Q and R are operators that are bounded and linear linked to the norm endowed with an operator G that is non-negative and that satisfy, $\|Q + \gamma R\|_G \geq \|Q\|_G$ for all $\gamma \in \mathbb{C}$. Zamani [11] extended the study due to Bhatia and Semrl, and proved that $Q \perp_G^B R$ provided that there exist a sequence of G-unit vectors $\{x_n\}$ in H such that $\lim_{n\to\infty} \|Qx_n\|_G = \|Q\|_G$ and $\lim\langle Qx_n, Rx_n\rangle_G = 0$. Then, formulas for the operator distance G to the set of multiple scalars in Semi-Hilbert spaces were provided. The following are Zamani's main results.

Theorem 2.97. *[11, Theorem 2.4] Suppose Z is a finite dimensional Hilbert space and let $Q, R \in L(Z)$ then the properties below are equivalent*

(i). There exists $z \in M_Q$ such that $Q_z \perp_A R_z$

(ii). $Q \perp_A^B R$

Theorem 2.97 shows that Q_z is orthogonal to R_z is equivalent to Q is orthogonal to R but the study was limited to characterization of finite elementary operators.

In [12] Arpita and Kallol studied orthogonality defined on arbitrary Banach spaces. Arbitrary Banach spaces were characterized and similar ones were obtained under other added properties. For a Hilbert space Z that is arbitrary, Arpita and Kallol [12] also studied orthogonality to a subspace of the set of linear operators $L(Z)$ and linked it to operator norm and the numerical radius norm. Birkhoff-James orthogonality was characterized for bounded linear operators on Banach spaces that are arbitrary to a subspace of the space of an operator. Their main goal was to investigate Birkhoff-James orthogonality for $Q \in L(Z, W)$ to the subspace $L(Z, W)$ for Banach spaces Z and W that are arbitrary. Arpita and Kallol [12] first characterized $Q \perp R$ whenever $Q \in L(Z, W)$ and R is a subspace of finite dimension of $L(Z, W)$ where W is a reflexive Banach space and Z is a Banach space of finite dimension. If W and Z were arbitrary Banach spaces and R an arbitrary subspace of $L(Z, W)$, then $Q \perp_B W$ under appropriate conditions. Arpita and Kallol [12] also characterized Birkhoff-James orthogonality of $Q \in L(Z, W)$ to a subspace of $L(Z)$ for a Hilbert space Z of infinite dimension . Later, it was discovered that the norm attainment set is more important determining Birkhoff-James orthogonality of operators. The following are some of their main results:

Theorem 2.98. *[12, Theorem 2.8] Let Z be a Hilbert space and $Q \in L(Z)$ such that $\|Q\| = 1$, $M_Q = S_{Z_0}$ where Z_0 is a subspace Z of finite dimension and $\|Q\|_{Z_0^\perp} < \|Q\|$. Then for the subspace R of $L(Z)$, $Q \perp_B R$ if $z_1, z_2, ...z_n \in M_Q$ and $\gamma_1, \gamma_2, ...\gamma_n > 0$ such that $\sum_{i=1}^{n} = 1$ and $\sum_{i=1}^{n} \gamma_i \langle A_{zi}, B_{zi} \rangle = 0$ for all $A \in W$.*

Theorem 2.98 gives a complex characterization of orthogonality for arbitrary Banach spaces but the study was limited to characterization of finite elementary operators.

In [88] Bottazi, Conde and Debmalya studied bounded linear operators through Birkhoff-James orthogonality and isosceles orthogonality on Hilbert spaces and Banach spaces. Birkhoff-James orthogonality of bounded linear operators was introduced and some of the possible application were determined to this regard. Isosceles orthogonality of bounded positive linear operators defined on a Hilbert space and some of the related properties that those with disjoint support were studied. Birkhoff orthogonality was related to isosceles orthogonality in a general Banach space, Birkhoff orthogonality and isosceles orthogonality and norm attainment set and disjoint support. The following are some of their main results:

Proposition 2.99. *[88, Proposition 3.2] Suppose Z and W are two Banach spaces, either both real or complex. Let $Q, R \in L(Z, W)$. If $O_{QR} = S_Z$ the $Q \perp_B R$.*

Proposition 2.99 shows how bounded linear operators are characterized through the orthogonality of Birkhoff but did not determine Birkhoff-James orthogonality for finite elementary operators.

Theorem 2.100. *[88, Theorem 3.3] Suppose Z is a reflexive real Banach space and W is a real Banach space and $Q, R \in K(Z, W)$. If $Q \perp_B R$ then $O_{QR} \neq 0$.*

Theorem 2.100 shows how bounded linear operators are characterized in terms of Birkhoff-James orthogonality but did not determine Birkhoff-James orthogonality for finite elementary operators.

Theorem 2.101. *[88, Theorem 3.4] A real or complex Hilbert space H is finite-dimensional if for any $Q, R \in K(Z, W)$ we have $Q \perp_B R \Rightarrow O_{QR} \neq 0$.*

Theorem 2.101 considered a complex Hilbert space and determined the orthogonality of Birkhoff of operators that are linear and bounded but did not determine Birkhoff-James orthogonality for finite elementary operators.

In [22] Bhuwan and Prakash enlisted some properties of Birkhoff-Orthogonality and Carlsson orthogonality along with it, Bhuwan and Prakash [22] introduced two new particular cases of Carlsson orthogonality and checked some properties of orthogonality in relation to these particular cases in normed spaces. Bhuwan and Prakash [22] showed how isosceles, Rorbert and Pythagorean orthogonalities can be derived from the carlsson orthogonality and obtained two new orthogonality relations for the Carlsson.

In [73] Priyanka and Sushil considered the required properties for the orthogonality of Birkhoff in Banach spaces. The relations between the notion of Gateaux derivative and orthogonality for the sub-differential set function of norm were given. Formulas for distances which can be obtained by the characterizing Birkhoff-James orthogonality were obtained. Finally, few new results were obtained. The following are some of their results:

Theorem 2.102. *[73, Theorem 1.1] If $Q, R \in M_n(K)$. Then Q is orthogonal to R if there exist a unit vector $r \in C^n$ and that $\|Q_r\| = \|Q\|$ and $\langle Q_r, R_r \rangle = 0$.*

Theorem 2.102 provides orthogonality properties for Birkhoff-James orthogonality in Banach space but the study was limited to establishment of orthogonality conditions for finite elementary operators.

Theorem 2.103. *[73, Theorem 3.2] Let Z be a complex Hilbert space. Let $Q, R \in L(Z)$. Then Q is orthogonal to R if there exist a sequence of* *unit vectors $h_n \in H$ such that $\|Qhn\| \to \|Q\|$ and $\langle Qhn, Rhn\rangle \to 0$ as $n \to \infty$.*

Theorem 2.103 provides required orthogonality conditions for Birkhoff-James orthogonality in complex Hilbert space but the study was limited to establishment of orthogonality conditions for finite elementary operators.

2.4 Orthogonality of elementary operators

Concerning elementary operators and their orthogonality Anderson [5] studied the range-kernel orthogonality for normal derivations. In his investigations Anderson [5] proved that if Q and R are operators in $L(Z)$ such that Q is normal and $QS = SQ$ then for very $Y \in L(Z)$, $\|\delta_Z(Y) + R\| \geq \|R|$ whereby $\|.\|$ is the usual operator norm. It was shown that if P is an isometry or a normal operator then the range of δ_T is orthogonal to its nullspace. Also Anderson [5] showed that if P is normal with an infinite number of points then the closed linear space of the range-kernel orthogonality of δ_P is not all of $L(Z)$. The following are Anderson's main results.

Theorem 2.104. *[5, Theorem 3.2] Let S be an isometry in $L(H)$. Then R is orthogonal to $N(\delta_S)$.*

Theorem 2.104 shows that the range of (δ_S) is orthogonal to the kernel of (δ_S) but the study was limited to establishment of orthogonality conditions for finite elementary operators.

Theorem 2.105. *[5] Let Q be normal in $L(H)$ with spectral measure $E(.)$. Then $\forall X \in L(H)$ and $\forall R \in Q(\delta_R)$, $\|R - \delta_Q(X)\| \geq \|R\|$ i.e $Ran(\delta_R)$ is orthogonal to $Ker(\delta_R)$.*

Theorem 2.105 considered a normal operator in $L(H)$ with spectral measure $E(.)$ and showed that the range of (δ_R) is orthogonal to kernel of (δ_R) but the study was limited to establishment of orthogonality conditions for finite elementary.

Kittaneh [65] extended the study and showed that if Q and R are operators in $L(H)$ and that Q is normal R is an operator and $R \in \{Q\}$ then for every $Y \in L(H)$, $\|\delta_Q(Y) + R\|_2^2 \geq \| \ \delta_Q(Y)\|_2^2 + \|R\|_2^2$ where $\|.\|_2$ is the Hilbert Schmidt operator norm. Therefore, in Hilbert space sense, the range of δ_Q of Hilbert Schmidt operators is orthogonal to the kernel of δ_Q.

In [66] Kittaneh used the p-norms of schatten for them being Gateaux differentiatiable and the operator norm to determine the range kernel orthogonality operators with respect to these norms. The following are some of Kittaneh's main results:

Theorem 2.106. *[66, Theorem 1] Let $A \in L(H)$ and $G \in C_P$ for some p with $1 < w \leq \infty$. Then $\|\delta_N(X) + G\|_w \geq \|G\|_w$ for every $X \in L(H)$ with $\delta_N \in C_P$ if $A\widetilde{G} = \widetilde{G}A$.*

Theorem 2.106 shows that the range of a normal derivation is orthogonal to its kernel but did not establish orthogonality conditions for finite elementary operators.

Theorem 2.107. *[66, Theorem 1] Let* $A \in L(H)$ *and* $S \in C_P$ *for some* p *with* $1 < p \leq \infty$. *Then* $\|\delta_N(Z) + S\|_p \geq \|S\|_p$ *for every* $Z \in L(H)$ *with* $\delta_N(Z) \in C_P$ *if* $tr(\widetilde{S}\delta_N(Z)) = 0$.

Theorem 2.107 shows that the range of $\delta_N(Z)$ is orthogonal to the kernel of $\delta_N(Z)$ but did not establish orthogonality conditions for finite elementary operators.

In [37] Duggal considered an elementary operator δ_{qh} where the operators q, h, r are hyponormal, the operators q_1, h_2 are normal and q_1 commutes with h_2. The following are Duggal's main results:

Theorem 2.108. *[37, Theorem 2.4] Let* $a, b \in L(Z)$ *be commuting normal operators and* $\phi : L(Z) \to L(Z)$ *be given by* $\phi(X) = qrq^* - hrh^*$, *then* $\|\phi(r) + s\|_\phi \geq \|s\|_\phi$ *for all* $s \in \phi^{-1}(0) \cap \phi$ *and all* $r \in L(Z)$.

Theorem 2.108 gives the range-kernel orthogonaliy of an elementary but did not establish orthogonality conditions for finite elementary operators.

In [89] Turnsek studied the elementary operator $\varphi; L(H) \to L(H)$ defined by $\varphi(Z) = \sum_{i=1}^{k} A_i Z B_i$ and $\varphi^*(Z) = \sum_{i=1}^{k} A_i^* Z B_i^*$. Tursek [89] proved that

(i). When $\varphi \leq 1$, then $\|\varphi(Z) - Z + G\| \geq \|G\|$ for all $Z \in L(H)$ and $G \in Ker\varphi$.

(ii). When $\sum_{i=1}^{k} AiAi^* \leq 1$, $\sum_{i=1}^{k} Ai^*Ai \leq 1$, $\sum_{i=1}^{k} BiBi^* \leq 1$ *and* $\sum_{i=1}^{k} B_i^* B_i \leq 1$ then for $G \in Ker\varphi \cap Ker\varphi^* \cap l_p$, $(1 \leq p \leq \infty)$ $\|\varphi Z - Z + G\|_p \geq \|G\|_p$ and $\|\varphi^* Z - Z + G\|_p \geq \|G\|_p$ for *every* $Z \in L(H)$

(iii). $(M_i)_{i=1}^{k}$ and $(N_i)_{i=1}^{k}$ be sequences of normal operators that commute and let $\delta(Z) = \sum_{i=1}^{k} M_i Z N_i$. If $\delta(Z) \in l_2$ and $G \in ker\delta \cap l_2$, then $\|\delta(Z) + G\|_2^2 = \|\delta(Z)\|_2^2 + \|G\|_2^2$

The following are some of his main results:

Theorem 2.109. *[89, Theorem 1.1] Let G be a normed algebra such that* $\|qr\| \leq \|q\|\|r\|$ *for all* $q,\ r \in G$ *and let* $\phi : G \to G$ *be a linear map with* $\phi \leq 1$. *If* $\phi(S) = S$ *for every* $S \in G$, *then* $\|\phi(Z) - Z + S\| \geq \|S\|$ *for all* $Z \in G$.

Theorem 2.109 considered a normed algebra and showed that $\phi - 1$ preserves orthogonality if $\phi \leq 1$ but did not determine Birkhoff-James orthogonality for finite operators.

Proposition 2.110. *[89, Proposition 1.2] Suppose that* $\phi(S) = S$ *for some* $S \in L(H)$. *If* $\phi \leq 1$, *then* $\|\phi(Z) - Z + S\| \geq \|S\|$.

Proposition 2.110 shows that for an elementary operator $\phi; L(H) \to L(H)$ preserves orthogonality if $\phi \leq 1$ but did not determine Birkhoff-James orthogonality for finite operators.

Dragoljub [27] gave the range-kernel orthogonality results of the elementary operator linked to the invariant norms that are unitarily related with the norm ideals of operators. The set involved the mapping $Q : L(Z) \to L(Z)$, $Q(X) : RXG + KXV$ where $L(Z)$ is the group of all bounded operators and $R,\ G,\ K,\ V$ are normal operators such that $RK = KR$, $GV = VG$ and $KerR \cap KerK = KerG \cap KerV = \{0\}$. Dragoljub [27] established this set in the sense that the widest set that the orthogonality result is valid. Dragoljub [27] obtained the following results:

Theorem 2.111. *[27, Theorem 1] Suppose* $Q, R \in L(H)$ *are normal operators such that* $QR = RQ$ *and* $\phi(X) = QXR - RXQ$*. Furthermore suppose that* $Q^*Q + R^*R > 0 \;\; if \;\; S \in Ker\phi \;\; then \;\; |||\phi(X) + K||| \geq |||K|||$.

Theorem 2.111 presents the result of orthogonality of elementary operators to an arbitrary unitary invariant norms but did not characterize finiteness of elementary operators.

Bachir and Hashem [17] presented a new set of finite operators which include the set of dominant operators and gave an extension of the orthogonality results to certain finite operators. In [17] some commutativity results were generalized. Their main goal was to investigate the orthogonality of Ran$\delta_{Q,G}$ and Ker$\delta_{Q,G}$ for certain finite operators. It was proved that $R(\delta_{Q,G})$ is orthogonal to Ker$\delta_{Q,G}$ when Q is dominant and G^* is M-hyponormal. The following are the main results obtained:

Proposition 2.112. *[17, Proposition 3.1] Let* Q *be dominant and* G *be a normal operator such that* $QG = GQ, \;\; then \;\; for \;\; all \;\; \lambda \in \delta_p(G), \;\; |\lambda| \leq dis(N, R(\delta_Q))$.

Proposition 2.112 considered a dominant operator and showed that $R(\delta_Q)$ is orthogonal to Kerδ_Q for every $\lambda \in \delta_p(Q)$ but did not establish orthogonality conditions for this finite operator.

Proposition 2.113. *[17, Proposition 3.3] Let* Q *be dominant then for every normal operator* G *such that* $QG = GQ$ *we have* $\|G\| \leq dist(G, R(\delta_Q))$.

Proposition 2.113 shows that $R(\delta_Q)$ is orthogonal to Kerδ_Q when Q is dominant for every normal operator G but did not characterize orthogonality conditions for finite operators.

Theorem 2.114. *[17, Theorem 3.4] Let Q be dominant and $G^\star$ is hyponormal, then for every $T \in Ker(\delta_Q)$ we have $\|T\| \leq dist(T, R(\delta_Q))$.*

Theorem 2.114 shows that $R(\delta_Q)$ is orthogonal to Kerδ_Q *when* A is dominant and G^* is M-hyponormal but did not characterize orthogonality conditions for these operators.

Duggal and Harste [38] studied orthogonality and properties for closre of the range for some elementary operators obtained from hyponormal operators or contractions on Hilbert spaces. The following are some of their main results:

Theorem 2.115. *[38, Theorem 1] Let $Q, G \in L(H)$ are contractions then $\|Z + D_{Q,G}(R)\| \geq \|Z\|\| - \sqrt{8\|D_{Q,G}(Z)\|\|R\|}$.*

Theorem 2.115 shows range closure properties for some elementary operators derived from hyponormal operators and establishes their orthogonality but did not establish orthogonality conditions for finite elementary operators.

Okelo and Agure [72] presented different examples of orthogonality in normed spaces and gave the range-kernel orthogonality results of elementary operators and the operators that implement them were then provided. The following are some of their results:

Theorem 2.116. *[72, Theorem 3.10] Let Y be an ideal with reflexivity in $L(Z)$ such that H^* is strictly convex and let $G : Y \to Y$ be the elementary* *operator defined by* $G(Z) = SZV + KZL$ *where* $S, V, K, L \in L(Z)$ *are operators that are normal and that* $SK = KS$, $VL = LV$, $SS^* \leq KK^*$, $VV^* \leq LL^*$. *Then* $H = \overline{RanG} \oplus KerG$.

Theorem 2.116 shows that the range of a Jordan elementary operator is orthogonal to its kernel but the study was limited to characterization of finiteness of elementary operators.

In [13] Bouali and Bouhafsi exhibited pair (Q, G) of operators and showed that the range of $\delta_{Q,G}$ is orthogonal to the kernel of $\delta_{Q,G}$ for the usual operator norm. In [13] range and kernel orthogonality for $\delta_{Q,G}$ in relation to the set of unitary invariant norms were established. The following are some of their main results.

Theorem 2.117. *[13,Theorem 2.1] Suppose* $Q, G \in L(Z)$. *If* G *is invertible and* $\|Q\|\|G^{-1}\| \leq 1$ *then* $\|\delta_{Q,G}(Z) + R\| \geq \|R\|$ $for\ \ all$ $Z \in L(Z)$ *and* $for\ \ all$ $R \in (Ker\delta_{Q,G})$.

Theorem 2.117 investigated the orthogonality of range $\delta_{Q,G}$ and kernel $\delta_{Q,G}$ of operators $Q, G \in L(Z)$ but did not establish orthogonality conditions for these operators.

Theorem 2.118. *[13,Theorem 2.2] Suppose* $Q, G \in L(Z)$. *If either*

(i). Q *is an isometry and the operator* G *is a contraction or*

(i). Q *is a contraction and* G *is a co-isometric then* $\|\delta_{Q,G}(Z) + R\| \geq \|R\|$ $for\ \ all$ $Z \in L(Z)$ *and* $for\ \ all$ $R \in (Ker\delta_{Q,G})$.

Theorem 2.118 investigated the orthogonality of range $\delta_{Q,G}$ and kernel $\delta_{Q,G}$ of operators $Q, G \in L(Z)$ on the condition that either Q is an isometry and the operator G is contractive or Q is contractive and G is a co-isometric but did not establish orthogonality conditions for these operators.

Bachir and Nawal [16] studied and characterized the points $C_1(H)$, the trace class operators that are orthogonal to the range of elementary operators in non-smoothness case and gave a counter example.

Theorem 2.119. *[16,Theorem 1] If Q is a non open linear subset of $r \notin Q$, then $r \perp s \Leftrightarrow \nexists \overline{\phi} \in D(r)$:$Q \subset ker\overline{\phi}$.*

Theorem 2.119 characterized the points $C_1(H)$, the trace class operators that are orthogonal to the range of elementary operators in non-smoothness case but did not determine Birkhoff-James orthogonality for finite elementary operators.

Corollary 2.120. *[16,Corollary 1] Let Q be a normal operator in $L(H)$ with $\delta_{pr}(Q) = \phi$ and $S \notin Ker\delta_Q$ be a positive operator in $C_1(H)$ then $\|\delta_Q(Z) + H\| \geq \|H\|$ for all $Z \in C_1(H)$.*

Corollary 2.120 characterized the points $C_1(H)$, the trace class operators that are orthogonal to the range of elementary operators in non-smoothness case but did not determine Birkhoff-James orthogonality for finite elementary operators.

In [70] Okelo characterized orthogonality of operators that are elementary in classes that attain norms. In [70] required conditions for norm attainability of Hilbert spaces operators were obtained. Okelo [70] then gave the range-kernel orthogonality results for elementary operators when they are generalized by norm-attainable operators in norm-attainable groups. The following are Okelo's main results:

Proposition 2.121. *[70, Proposition 3.1] Suppose* $Q, R, Z \in \Omega$ *and* $ZR = I$ *where* I *is an identity element of* Ω*. Then for* $\delta_{Q,R} = QX - XR$ *and an elementary operator* $\theta_{Q,R}(X) = QXR - XR_Q\overline{(Ran(\delta_{Q,R}))} \cap Ker(\delta_{Q,Z}) = \overline{Ran\theta_{Q,R}} \cap Ker\theta_{Q,R}$.

Proposition 2.121 gives the range-kernel orthogonality results for elementary operator so but did not characterize finiteness of these elementary operators. In our study we characterized finiteness of elementary operators.

Theorem 2.122. *[70, Theorem 3.10] Suppose* $Q,\ R,\ S,\ T \in NAH$ *are normal operators such that* $QS = SQ,\ RT = TR,\ QQ^* \leq SS^*,\ RR^* \leq TT^*$*. For an elementary operator* $\phi(X) = QXR - SXT$ *and* $G \in NA(H)$ *satisfying* $Q \leq R = S \leq T$ *then* $\|\phi(X)+G\| \geq \|G\|$ *for all* $X \in NA(H)$.

Theorem 2.122 shows that the range of a Jordan elementary operator is orthogonal to its kernel in norm-attainable classes but did not characterize finiteness of these elementary operators.

2.5 Summary of gaps

Williams [92] showed that the group of finite operators is uniformly closed, involves normal operators, paranormal operators, operators that have completely continuous direct summand, and every C^* algebra but the study did not give a detailed description of finite operators. In our study we gave a detailed description of finite operators and established orthogonality conditions for these operators. Okelo and Agure [72] presented various notions of orthogonality in normed spaces and gave the range-kernel orthogonality results for elementary operators, they established orthogonality conditions for elementary operators but did not establish orthogonality conditions when elementary operators are finite. In our study, therefore we considered finiteness of elementary operators and established orthogonality conditions for finite elementary operators in terms of James-Birkhoff orthogonality.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

In this chapter, we discuss the methods that we used to successfully achieve our objectives. The methodology involved the use of Gram Schmidt procedure, Berberian technique, Fuglede Putnam property, use of known inequalities such as Cauchy Schwarz inequality, triangle inequality, Hölder's inequality, Minkowski's inequality and Bessel's inequality. We also used technical approaches of tensor products and direct sum decomposition.

3.2 Known inequalities

In this section, we discuss the known inequalities and they include Cauchy Schwarz inequality, triangle inequality, Hölder's inequality, Minkowski's inequality and Bessel's inequality.

3.2.1 Cauchy-Schwarz inequality

Let $(Y, \langle ., . \rangle)$ be an inner product space over a field $\mathbb{F}$ then for all $s, p \in Y$ $|\langle s, p\rangle| \leq \sqrt{\langle s, s\rangle}\sqrt{\langle p, p\rangle}$. Moreover, given any $s, p \in Y$, the inequality $|\langle s, p\rangle| \leq \sqrt{\langle s, s\rangle}\sqrt{\langle p, p\rangle}$ holds if and only if $s \quad and \quad p$ are linearly dependent. Indeed, if $s = 0$ or $p = 0$, then the inequality holds. Assume that $s \neq p$ and $p \neq 0$. For any $\eta \in \mathbb{F}$, we have $0 \leq \langle s - \eta p, s - \eta p\rangle = \langle s, s\rangle$ - $\overline{\eta}\langle s, p\rangle$-$\eta\langle p, s\rangle$-$\eta\overline{\eta}\langle p, p\rangle$. Now choosing $\eta = \frac{\langle s,s\rangle}{\langle p,p\rangle}$, we have $0 \leq \langle s, s\rangle - \frac{|\langle s,p\rangle|^2}{\langle p,p\rangle} - \frac{|\langle s,p\rangle|^2}{\langle p,p\rangle} + \frac{|\langle s,p\rangle|^2}{\langle p,p\rangle}$=$\langle s, s\rangle - \frac{|\langle s,p\rangle|^2}{\langle p,p\rangle}$. Hence, $|\langle s, p\rangle| \leq \sqrt{\langle s, s\rangle}\sqrt{\langle p, p\rangle}$. Assume that $|\langle s, p\rangle| \leq \sqrt{\langle s, s\rangle}\sqrt{\langle p, p\rangle}$. We show that s and p are linearly dependent. If $s = 0$ and $p = 0$, then s and p are obviously linearly dependent. We hence assume that $s \neq 0$ and $p \neq 0$. Then $\langle p, p\rangle \neq 0$ with $\eta = \frac{\langle s,s\rangle}{\langle p,p\rangle}$, we have that $\langle s - \eta p, s - \eta p\rangle$ $= \langle s, s\rangle - \frac{|\langle s,p\rangle|^2}{\langle p,p\rangle} = 0$. That is, $\langle s - \eta p, s - \eta p\rangle = 0 \Rightarrow s = \eta p$. That is, s and p are linearly dependent.

3.2.2 Triangle inequality

For all $s, p \in \mathbb{R}$, $\|s + p\| \leq \|s\| + \|p\|$. Indeed, we have that

$$\begin{aligned}
\|s + p\|^2 &= \langle s + p, s + p\rangle \\
&= \|s\|^2 + \langle s, p\rangle + \langle p, s\rangle + \|p\|^2 \\
&= \|s\|^2 + 2Re\langle s, p\rangle + \|p\|^2 \\
&\leq \|s\|^2 + 2|\langle s, p\rangle| + \|p\|^2 \\
&\leq \|s\|^2 + 2\|s\|\|p\| + \|p\|^2 \\
&= (\|s\| + \|p\|)^2
\end{aligned}$$

Taking square roots gives the triangle inequality i.e $\|s+p| \leq \|s\| + \|p\|$.

3.2.3 Hölder's inequality for sequences

Let $(s_n) \in l_p$ and $(t_n) \in l_q$, where $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^{\infty} |s_k t_k| \leq (\sum_{k=1}^{\infty} |s_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |t_k|^q)^{\frac{1}{q}}.$$

Indeed, if $\sum_{k=1}^{\infty} |s_k|^p = 0$ or $\sum_{k=1}^{\infty} |t_k|^p = o$. Then the inequality holds. Assume that $\sum_{k=1}^{\infty} |s_k|^p \neq 0$ and $\sum_{k=1}^{\infty} |t_k|^q \neq 0$. Then for $k = 1, 2, ..$ by Young's inequality we have,

$$\frac{|s_k||t_k|}{(\sum_{k=1}^{\infty} |s_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |t_k|^q)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{|s_k|^p}{\sum_{k=1}^{\infty} |s_k|^p} + \frac{1}{q} \frac{|t_k|^q}{\sum_{k=1}^{\infty} |t_k|^q}.$$

Hence

$$\frac{\sum_{k=1}^{\infty} |s_k t_k|}{(\sum_{k=1}^{\infty} |s_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |t_k|^q)^{\frac{1}{q}}} \leq \frac{1}{p} + \frac{1}{q} = 1.$$

This implies that

$$\sum_{k=1}^{\infty} |s_k t_k| \leq (\sum_{k=1}^{\infty} |s_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |t_k|^q)^{\frac{1}{q}}.$$

3.2.4 Minkowski's inequality for sequences

Let $p > 1$ and $(g_n), (h_n)$ be sequences in l_p. Then

$$(\sum_{k=1}^{\infty} |g_k + h_k|^p)^{\frac{1}{p}} \leq (\sum_{k=1}^{\infty} |g_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |h_k|^p)^{\frac{1}{p}}.$$

Indeed, let $q = \frac{p}{p-1}$. If $\sum_{k=1}^{\infty} |g_k + h_k|^p = 0$, then the inequality holds. We therefore assume that $\sum_{k=1}^{\infty} |g_k + h_k|^p \neq 0$. Then we have,

$$\begin{aligned}
\sum_{k=1}^{\infty} |g_k + h_k|^p &= \sum_{k=1}^{\infty} |g_k + h_k|^{p-1} |g_k + h_k| \\
&\leq \sum_{k=1}^{\infty} |g_k + h_k|^{p-1} |g_k| + \sum_{k=1}^{\infty} |g_k + h_k|^{p-1} |h_k| \\
&\leq (\sum_{k=1}^{\infty} |g_k + h_k|^{(p-1)q})^{\frac{1}{q}} [(\sum_{k=1}^{\infty} |g_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |h_k|^p)^{\frac{1}{p}}] \\
&= (\sum_{k=1}^{\infty} |g_k + h_k|^p)^{\frac{1}{q}} [(\sum_{k=1}^{\infty} |g_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |h_k|^p)^{\frac{1}{p}}].
\end{aligned}$$

Dividing both sides by $(\sum_{k=1}^{\infty} |g_k + h_k|^p)^{\frac{1}{q}}$ we have

$$(\sum_{k=1}^{\infty} |g_k + h_k|^p)^{\frac{1}{p}} = (\sum_{k=1}^{\infty} |g_k + h_k|^p)^{1-\frac{1}{p}} \leq (\sum_{k=1}^{\infty} |g_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |h_k|^p)^{\frac{1}{p}}.$$

3.2.5 Bessel's inequality

Let P be an orthonormal set in an inner product space Y. Let $q_1, q_2, ...q_n$ be a finite subset of P, for all $r \in Y$. Then $\sum_{i=1}^{n} |\langle r, q_i \rangle|^2 \leq \|r\|^2$.

Indeed, we show that $0 \leq \|r - \sum_{i=1}^{n} \langle r, q_i \rangle q_i\|^2$ Let $\alpha_i = \langle r, q_i \rangle$ and $\alpha_j =$

$\langle r, q_j \rangle$. Then we have,

$$\begin{aligned}
0 &\leq \|r - \sum_{i=1}^{n} \langle r, q_i \rangle q_i\|^2 \\
&\leq \langle r - \sum_{i=1}^{n} \alpha_i q_i, r - \sum_{i=1}^{n} \alpha_i q_i \rangle \\
&\leq \langle r, r \rangle - \langle r, \sum_{i=1}^{n} \alpha_i q_i \rangle - \langle \sum_{i=1}^{n} \alpha_i q_i, r \rangle + \langle \sum_{i=1}^{n} \alpha_i q_i, \sum_{i=1}^{n} \alpha_i q_i \rangle \\
&= \|r\|^2 - \sum_{i=1}^{n} \overline{\alpha_i} \langle r, q_i \rangle - \sum_{i=1}^{n} \alpha_i \langle q_i, r \rangle + \sum_{i=1}^{n} \sum_{i=1}^{n} \alpha_i \alpha_i \langle q_i, q_i \rangle but \langle q_i, q_i \rangle = 1 \\
&= \|r\|^2 - \sum_{i=1}^{n} \overline{\alpha_i} \alpha_i + \sum_{i=1}^{n} \alpha_i \overline{\alpha_i} + \sum_{i=1}^{n} |\alpha_i|^2 \\
&= \|r\|^2 - \sum_{i=1}^{n} |\alpha_i|^2.
\end{aligned}$$

That is $\sum_{i=1}^{n} |\alpha_i|^2 = \|r\|^2$.

This implies that $\sum_{i=1}^{n} |\langle r, q_i \rangle|^2 \leq \|r\|^2$.

3.2.6 Gram Schmidt procedure

If $\{k_1, k_2, ...k_p\}$ is a set that is linearly independent in an inner product space X then there exist an orthogonal set $\{j_1, j_2, ...j_p\}$ in X such that span $\{k_1, k_2, ...k_i\}$ = span$\{k_1, k_2, ...k_i\}$ for $(i = 1, ..., p)$.

$j_1 = k_1$.

$j_2 = k_2 - \frac{\langle k_2, j_1 \rangle}{\langle j_1, j_1 \rangle} j_1$.

$j_3 = k_3 - \frac{\langle k_3, j_1 \rangle}{\langle j_1, j_1 \rangle} j_1 - \frac{\langle k_3, j_2 \rangle}{\langle j_2, j_2 \rangle} j_2$.

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$j_p = k_p - \frac{\langle k_p, j_1 \rangle}{\langle j_1, j_1 \rangle} j_1 - \frac{\langle k_p, j_2 \rangle}{\langle j_2, j_2 \rangle} j_2, \ldots \frac{\langle k_p, j_{p-1} \rangle}{\langle j_{p-1}, j_{p-1} \rangle} j_{p-1}.$

Then $j_1, j_2, \ldots j_n$ is an orthogonal basis for j.

Normalizing each j_j results in an orthonormal basis. That is $u_j = \frac{j_i}{\|j_i\|}$.

3.3 Technical approaches

In this section we discuss technical approaches and they include Direct sum decomosition and tensor products.

3.3.1 Direct sum decomposition

Let Q and R be subspaces of Z. Then Z is said to be the direct sum of Q and R if $Z = Q + R$ and $Q \cap R = 0$ and we write $Z = Q \oplus R$.

3.3.2 Tensor product

Let R and S be vector spaces over $\mathbb{K}$ and let Q be the subspace of the free vector space $\mathbb{K}_{R \times S}$ generated by all the vectors of the form;

$$\alpha(r, s) + \beta(r^{`}, s) - (\alpha g + \beta g^{`}, s)$$

and

$$\alpha(r, s) + \beta(g, h^{`}) - (r, \alpha s + \beta s^{`}),$$

$\forall \alpha, \beta \in \mathbb{K}$ and $r, r` \in X,\ s, s` \in Y$. Then the quotient space $\mathbb{K}_{(R\times S)/Q}$ is called the tensor product of R and S denoted by $R \otimes S$.

3.3.3 Berberian technique

Proposition 3.1. *Let Y be a complex Hilbert space, then there is a Hilbert space $\hat{Y} \supset Y$ and $\psi : L(Y) \to L(\hat{Y})$ $(P \to \hat{P})$ where ψ is a $*-$isometric-isomorphism satisfying the order such that:*

(1) $\psi(J^*) = \psi(J)^*$;

(2) $\psi(I) = \hat{I}$;

(3) $\psi(\alpha J + \beta Q) = \alpha\psi(J) + \beta\psi(Q)$;

(4) $\psi(JQ) = \psi(J) + \psi(Q)$;

(5) $\|\psi(J)\| = \|J\|$;

(6) $\psi(J) \leq \psi(Q)$ *if* $J \leq Q$, $\forall J, Q \in L(Y)$, $\alpha, \beta \in \mathbb{C}$;

(7) $\sigma(J) = \sigma(\hat{J})$, $\sigma_a(J) = \sigma_a(\hat{J}) = \sigma_p(\hat{J})$.

3.3.4 Putnam-Fuglede property

Let $Q, P \in L(H)$ be normal operators, then the pair (P, Q) of operators has the following Putnam-Fuglede property(PF):
If $PX = XQ$ where $X \in L(H)$, then $P^*X = XQ^*$.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, we give the results of our study. We consider finiteness of elementary operators, orthogonality conditions for finite elementary operators and Birkhoff-James orthogonality for finite elementary operators.

4.2 Finiteness of elementary operators

Proposition 4.1. *Let Ω be a normed space, then for $S \in \Omega$, $\sigma_p(S) \neq \emptyset$ if S is normaloid.*

Proof. Let $S \in \Omega$ be normaloid, then $\|S\| = r(S)$. This means that there exist $\lambda \in \sigma_p(S)$ such that $|\lambda| = \|S\|$. It is known that $\sigma_p(S) \subseteq \sigma_{ap}(S) \subseteq \sigma(S)$. Therefore, $\sigma_p(S) = \sigma_{ap}(S)$. But λ is in the boundary of $\sigma_p(S)$ and since this is a subset of the approximate point spectrum of S, we have that $\lambda \in \sigma_p(S) = \sigma_{ap}(S)$. But for a sequence $\{x_n\}_{n\in N}$ of unit vectors we have, $\|(S - \lambda I)x_n\| \to 0$. So $0 \in \sigma_p(S)$ and hence $\sigma_p(S) \neq \emptyset$. □

Proposition 4.2. *Every normaloid operator is finite.*

Proof. From Proposition 4.1 we have that $\sigma_p(S) \neq \emptyset$ if S is normaloid. To show that every normaloid operator is finite, we let S to be a normaloid operator, i.e $\|S\| = r(S)$. Hence, there exist $\lambda \in \sigma_p(S)$ such that $|\lambda| = \|S\|$. By definition, an operator S in a normed space Ω is finite if $\|SX - XS - I\| \geq 1$, for all $X \in \Omega$. But $\|(S - \lambda I)x_n\| \to 0$ with $\|x_n\| = 1$. From Gram schmidt procedure $\{x_n\}$ is a normalized sequence and hence we have,

$$\begin{aligned} \|(SX - XS) - I\| &= \|((S - \lambda I)X - X(S - \lambda I)) - I\| \\ &\geq |\langle (S - \lambda I)X_{x_n,x_n}\rangle - \langle X(S - \lambda I)_{x_n,x_n}\rangle - I| \\ &\geq |\langle (S - \lambda I)X - X(S - \lambda I)\rangle_{x_n,x_n} - I| \\ &\geq |\langle (SX - XS)_{x_n,x_n}\rangle - I|. \end{aligned}$$

Letting $n \to \infty$ we obtain $\|(SX - XS) - I\| \geq 1$. □

Lemma 4.3. *Let $S \in \Omega$ be normaloid and $S_o \in \Omega$ be norm-attainable such that $SS_o = S_oS$. Then for every $\eta \in \sigma_p(S_o)$, $\|S_o - (SX - XS)\| \geq |\eta|$ $\forall X \in \Omega$.*

Proof. From [71], if $S_o \in \Omega$ is norm-attainable, then it is normal. So, we let $\eta \in \sigma_p(S_o)$ and M_η be the eigenspace associated with η. Because $SS_o = S_oS$, we have $SS_o^* = S_o^*S$ by Fuglede Putnam's theorem [41]. Hence M_η

reduces both S and S_o. Using the decomposition of $H = M_\eta \oplus M_\eta^\perp$, we define S, S_o and X as follows:

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix}, \; S_o = \begin{pmatrix} \eta & 0 \\ 0 & S_2 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

We have,

$$\begin{aligned} \|S_o - (SX - XS)\| &= \left\| \begin{pmatrix} \eta - (S_1X_1 - X_1S_1) & * \\ * & * \end{pmatrix} \right\| \\ &\geq \|\eta - (S_1X_1 - X_1S_1)\| \\ &\geq |\eta| \left\| 1 - \left(\left(\frac{S_1X_1}{\eta} \right) - \left(\frac{X_1S_1}{\eta} \right) \right) \right\| \\ &\geq |\eta|. \end{aligned}$$

□

Lemma 4.4. *Every paranormal operator in a unital C^* algebra Ω is finite.*

Proof. Let S be a paranormal operator, then S is normal i.e $S^*S = SS^*$. By Berberian theorem 3.3.3, we have that, there exist a $*$-isometric isomorphism $\psi : \Omega \to \Omega$ that preserves order such that,

$$\begin{aligned} \|S\|^2 = \|SS^*\| = 1 &\leq \|(SX - XS) - I\| \\ &\leq \|\psi(SX - XS) - I\| \\ &\leq \|(\psi(S)\psi(X) - \psi(X)\psi(S)) - I\|. \end{aligned}$$

If $S \in \Omega$ is an element of $F(H)$ such that $\sigma_p(S) \neq \emptyset$ then it results from Proposition 4.2 that $\psi(S) \in \Omega$ is finite i.e.

$$\|(SX - XS) - I\| = \|(\psi(S)\psi(X) - \psi(X)\psi(S)) - I\| \geq 1.$$

□

Theorem 4.5. *Let $S \in \Omega$ be norm-attainable. Then $J = S + P$ is finite where P is compact in a C^*-algebra Ω.*

Proof. Let S be norm-attainable, since Ω is a unital C^*-algebra, it follows that $J = S + P$ is finite. Indeed from Lemma 4.4 and Proposition 4.2 we have,

$$\begin{aligned} \|J\|^2 = \|JJ^*\| = 1 &\leq \|(JX - XJ) - I\| \\ &\leq \|(SX - XS) - I\| \\ &\leq \|(SX + PP^{-1} - XS + P^{-1}P) - I\| \\ &\leq \|(S + P)(X + P^{-1}) - (X + P^{-1})(S + P) - I\|. \end{aligned}$$

For $Y = X + P^{-1}$ we have, $\|(S+P)Y - Y(S+P) - I\| \geq 1$. This proves that $J = S + P$ is a finite operator. □

Corollary 4.6. *Let $S \in \Omega$ be log-hyponormal and S^*be p-hyponormal then $\|J - (SX - XS_o)\| \geq \|J\|$, for all $X \in \Omega$ and for all $J \in ker\delta_{S,S_o}$.*

Proof. If $J \in$Kerδ_{S,S_o}, then also $J \in$Ker$\delta_{S^*S_o^*}$ by Putnam-Fuglede's theorem [41]. Therefore, $SJJ^* = JS_o^* = JJ^*S$. Since S is log-hyponormal, JJ^* is normal and $S(JJ^*) = (JJ^*)S$. Since $X \in \Omega$, we deduce that

$$\begin{aligned} \|J\|^2 = \|JJ^*\| &= \|JJ^* - SXJ^* - XJ^*S\| \\ &\leq \|JJ^* - SXJ^* - XS_oJ^*\| \\ &\leq \|J^*\|\|J - (SX - XS_o)\| \end{aligned}$$

By Cauchy-Schwarz inequality 3.2.1, $\|J\|^2 = \|J\|\|J^*\|$.

This implies that $\|J\|^2 = \|J\|\|J^*\| \leq \|J^*\|\|J - (SX - XS_o)\|$.

Dividing both sides by $\|J^*\|$ we obtain,

$\|J\| \leq \|J - (SX - XS_o)\|$. $\square$

Remark 4.7. At this point, we characterize finiteness of elementary operators in a general set up. Let $\mathfrak{C}_n(S, S_o)$ be the set of all $(S, S_o) \in \Omega \times \Omega$ such that S and S_o have an n-dimensional reducing subspace $J_n(S, S_o)$ satisfying $S \mid J_n(S, S_o) = S_o \mid J_n(S, S_o)$.

Now, we characterize finiteness in the cartesian product of $\Omega \times \Omega$ in the next proposition.

Proposition 4.8. *Let $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. Then, the following inequality holds i.e $\|(SX - XS_o) - I\| \geq 1$.*

Proof. Let $\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$ be the matrix representation of S and S_o respectively relative to the decomposition $H = H_1 \oplus H_1^{\perp}$ where H_1 is an n-dimensional reducing subspace of S and S_o i.e $H_1 = J_n(S, S_o)$. For any operator X on H has a representation $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$. Let

$I = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$. It follows that,

$$\begin{aligned}
\|(SX - XS_o) - I\| &= \left\| \left[\begin{pmatrix} S_1X_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} S_1X_1 - X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} (S_1X_1 - X_1S_2) - I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&\geq \|(S_1X_1 - X_1S_2) - I_1\|.
\end{aligned}$$

This implies that

$$\|(SX - XS_o) - I\| \geq \|(S_1X_1 - X_1S_2) - I_1\| \geq \|I_1\| = \|I\|.$$

Hence, $\|(SX - XS_o) - I\| \geq 1$. □

Proposition 4.9. *Let $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. Then the following inequality holds i.e $\|(SXS_o) - I\| \geq 1$.*

Proof. Let S, S_o, X and I have the following representation:
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$, and $I = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$

From Proposition 4.8, it follows that,

$$\begin{aligned}
\|(SXS_o) - I\| &= \left\| \left[\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} S_1X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} (S_1X_1S_2) - I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|. \\
&\geq \|(S_1X_1S_2) - I_1\|.
\end{aligned}$$

This implies that

$$\|(SXS_o) - I\| \geq \|(S_1X_1S_2) - I_1\| \geq \|I_1\| = \|I\|.$$

Hence, $\|(SXS_o) - I\| \geq 1$. □

Theorem 4.10. *Let* $(S, S_o) \in \mathfrak{C}_n(S, S_o)$*. Then the following inequality holds i.e* $\|(SXS_o + S_oXS) - I\| \geq 1$.

Proof. Let S, S_o, X, and I have the following representation[decomposition].
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$, and $I = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$.

From Proposition 4.8 and Proposition 4.9 we have,

$$\begin{aligned}
\|(SXS_o + S_oXS) - I\| &= \left\| \left[\begin{pmatrix} S_1X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} S_2X_1S_1 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} S_1X_1S_2 + S_2X_1S_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} (S_1X_1S_2 + S_2X_1S_1) - I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&\geq \|(S_1X_1S_2 + S_2X_1S_1) - I_1\|.
\end{aligned}$$

This implies that

$$\|(SXS_o + S_oXS) - I\| \geq \|(S_1X_1S_2 + S_2X_1S_1) - I_1\| \geq \|I_1\| = \|I\|.$$

Hence, $\|(SXS_o + S_oXS) - I\| \geq 1$. □

Theorem 4.11. *Let $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. Then the following inequality holds i.e $\|(SXS_o + CXC_o) - I\| \geq 1$.*

Proof. Let S, S_o, C, C_o, X, and I have the following representation[decomposition].
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$, $S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}$,
$C_o = \begin{pmatrix} C_2 & 0 \\ 0 & 0 \end{pmatrix}$ and $I = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$.

From Theorem 4.10 we have,

$$\begin{aligned}\|(SXS_o + CXC_o) - I\| &= \left\| \left[\begin{pmatrix} S_1X_1S_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} C_1X_1C_2 & 0 \\ 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} S_1X_1S_2 + C_1X_1C_2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} (S_1X_1S_2 + C_1X_1C_2) - I_1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\ &\geq \|(S_1X_1S_2 + C_1X_1C_2) - I_1\|.\end{aligned}$$

This implies that

$$\|(SXS_o + CXC_o) - I\| \geq \|(S_1X_1S_2 + C_1X_1C_2) - I_1\| \geq \|I_1\| = \|I\|.$$

Hence, $\|(SXS_o + CXC_o) - I\| \geq 1$. □

Theorem 4.12. *Let* $(S, S_o) \in \mathfrak{C}_n(S, S_o)$. *Then the following inequality holds i.e* $\| \sum_{i=1}^{n} S_iXC_i - I\| \geq 1$.

Proof. Let S_i, X, C_i and I have the following representation.
$S_i = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$, $C_i = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}$, $I = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}$.

From Theorem 4.11 it follows that,

$$\begin{aligned}
\left\|\sum_{i=1}^{n} S_iXC_i - I\right\| &= \left\|\sum_{i=1}^{n}\left[\begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}\right] - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}\right\| \\
&= \left\|\sum_{i=1}^{n}\begin{pmatrix} S_1X_1C_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}\right\| \\
&= \left\|\sum_{i=1}^{n}\begin{pmatrix} (S_1X_1C_1) - I_1 & 0 \\ 0 & 0 \end{pmatrix}\right\| \\
&\geq \left\|\sum_{i=1}^{n}(S_1X_1C_1) - I_1\right\|.
\end{aligned}$$

This implies that

$$\|\sum_{i=1}^{n} S_iXC_i - I\| \geq \|\sum_{i=1}^{n}(S_1X_1C_1) - I_1\| \geq \|I_1\| = \|I\|.$$

Hence, $\|\sum_{i=1}^{n} S_iXC_i - I\| \geq 1$. □

Remark 4.13. It is known [78] that there is a compact operator C and that $R(\delta_c) = K(H)$. As a result we have that the $\text{dist}(I, K(H)) = 1$, where $\text{dist}(I, K(H))$ is the distance from I to $K(H)$. Hence, if S, S_o are compact operators, then $\text{dist}(I, R(\delta_{S,S_o})) = 1$.

4.3 Orthogonality conditions for finite elementary operators

In this section, we characterize orthogonality conditions for finite elementary operators. Let Ω denote a Banach algebra that is complex with identity I and let $\sigma_r(\Omega)$, $\sigma_l(\Omega)$ denote, respectively the right spectrum and the left spectrum of Ω. From [13]

$S^nX - XS^n = \sum_{i=0}^{n-i-1} S^{n-i-1}(SX - XS)S^i$ for all $X \in \Omega$.

If $SJ = JS$ we have,

$nJS^{n-1} = S^nX - XS^n - \sum_{i=0}^{n-i-1} S^{n-i-1}((SX - XS) - J)S^i$ for all $X \in \Omega$.

Proposition 4.14. *Let $S \in \Omega$, x_n be an increasing sequence of positive integers and S^{x_n} converge to $Z \in \Omega$, with $0 \notin \sigma_r(Z) \cap \sigma_l(Z)$. If there exist a constant λ such that $\|S^n\| \leq \lambda$ for all integers n and if S_o is the left or right inverse of Z then*

$$\lambda^2 \|S_o\| \|(SX - XS) - J\| \geq \|J\| \text{ for all } X \in \Omega \text{ and for all } J \in Ker\delta_S.$$

Proof. Let $X \in \Omega$, since

$nJS^{n-1} = S^nX - XS^n - \sum_{i=0}^{n-i-1} S^{n-i-1}((SX - XS) - J)S^i$ for $SJ = JS$.

We can write

$$\begin{aligned} (x_n + 1)JS^{x_n+1-1} &= S^{x_n+1}X - XS^{x_n+1} - \sum_{i=0}^{x_n+1-i-1} S^{x_n+1-i-1}((SX - XS) - J)S^i \\ &= S^{x_n+1}X - XS^{x_n+1} - \sum_{i=0}^{x_n-i} S^{x_n-i}((SX - XS) - J)S^i. \end{aligned}$$

when both sides are divided by $x_n + 1$ and if we take the norms we have,
$\|JS^{x_n}\| \leq \frac{1}{x_n+1} \| |S^{x_n+1}| + |S^{x_n+1}| \| \|X\| + \frac{1}{x_n+1} \sum_{i=0}^{x_n-i} \|S^{x_n-i}\| \|(SX - XS) - J\| \|S^i\|$

Since $\|S^n\| \leq \lambda$ we have that $\|S^{x_n+1}\| \leq \lambda$ and hence we obtain,

$$\|JS^{x_n}\| \leq \frac{2\lambda}{x_n+1}\|X\| + \lambda^2\|(SX - XS) - J\|.$$

Letting $n \to \infty$ we obtain,

$$\|JS^{x_n}\| \leq \lambda^2\|(SX - XS) - J\|.$$

But S^{x_n} converges to Z, so we have,

$$\|JZ\| \leq \lambda^2\|(SX - XS) - J\|.$$

Now, since S_o is in the right or the left of Z we have,

$$\|J\| \leq \|S_o\|\lambda^2\|(SX - XS) - J\|.$$

□

Remark 4.15. Let $S \in L(H)$ and x_n be a sequence that is increasing of integers that are positive. Assume that there is a constant λ and that $\|S^n\| \leq \lambda$ for all integers n

(i) . If $S^{x_n} \to P$, with $0 \notin \sigma_r(P) \cap \sigma_l(P)$, then $\lambda^2\|(SX - XS) - J\| \geq \|J\|$ for all $X \in L(H)$ and for all $J \in Ker\delta_S$.

(ii) . If $S^{x_n} \to P + K$, with K compact and $0 \notin \sigma_r(P) \cap \sigma_l(P)$, then $\lambda^2\|(SX - XS) - J - K\| \geq \|J\|$ for all $X \in L(H)$ and for all $J \in Ker\delta_S$.

Theorem 4.16. *Let $S \in L(H)$ such that $S^n = I$ for some integer n. Then $\lambda^2\|(SX - XS) - J\| \geq \|J\|$ for all $X \in L(H)$ and for all $J \in Ker\delta_S$.*

Proof. Since $S^n = \{I, S, S^2, ...S^{m-1}\}$ for all integers n, $\|S^n\| \leq \lambda$, $n \in \mathbb{N}$ and $S^{x_n} = I$, where $x_n = nm$, $n \in \mathbb{N}$. It is known that [13]

$nJS^{n-1} = S^nX - XS^n - \sum_{i=0}^{n-i-1} S^{n-i-1}((SX - XS) - J)S^i$, for all $X \in L(H)$.

From Proposition 4.14 we have that

$$(x_n + 1)JS^{x_n+1-1} = S^{x_n+1}X - XS^{x_n+1} - \sum_{i=0}^{x_n+1-i-1} S^{x_n+1-i-1}(SX - XS - J)S^i.$$

$$(x_n + 1)JS^{x_n} = S^{x_n+1}X - XS^{x_n+1} - \sum_{i=0}^{x_n-i} S^{x_n-i}((SX - XS) - J)S^i.$$

When both sides are divided by $x_n + 1$ and if we take the norms we have,
$\|JS^{x_n}\| \leq \frac{1}{x_n+1}\||S^{x_n+1}|+|S^{x_n+1}|\|\|X\|+\frac{1}{x_n+1}\sum_{i=0}^{x_n-i}\|S^{x_n-i}\|\|(SX-XS)-J\|\|S^i\|$

Since $\|S^n\| \leq \lambda$ we have that $\|S^{x_n+1}\| \leq \lambda$ and hence we obtain,

$$\|JS^{x_n}\| \leq \frac{2\lambda}{x_n+1}\|X\| + \lambda^2\|(SX - XS) - J\|.$$

Since $S^{x_n} = I$ we have,

$$\|J\| \leq \frac{2\lambda}{x_n+1}\|X\| + \lambda^2\|(SX - XS) - J\|.$$

Letting n tend to infinity, we get

$$\|J\| \leq \lambda^2\|(SX - XS) - J\|.$$

Hence, $\|J\| \leq \lambda^2 \|(SX - XS) - J\|$. $\square$

Corollary 4.17. *Let $S_1, S_o \in L(H)$ such that $S_1^m = I$ and $S_o^m = I$ for some integer m. Then*

$$\|(S_1X - XS_o) - J\| \geq \|J\| \text{ for all } X \in L(H) \text{ and for all } J \in Ker\delta_{S_1,S_o}.$$

Proof. Consider the operators P, S and Y defined on $H \oplus H$.
$P = \begin{pmatrix} S_1 & 0 \\ 0 & S_o \end{pmatrix}$, $S = \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$.
Then P is normal on $H \oplus H$ and that $P^m = 1$, $PS = SP$ i.e $S \in Ker\delta_p$.
Since $PY - YP = \begin{pmatrix} 0 & S_1X \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & XS_o \\ 0 & 0 \end{pmatrix}$

$$\begin{aligned} \|(PY - YP) - S\| &= \left\| \begin{pmatrix} 0 & S_1X - XS_o \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} 0 & (S_1X - XS_o) - J \\ 0 & 0 \end{pmatrix} \right\|. \end{aligned}$$

Then, it follows that

$$\|PY - YP - S\| \geq \|S\|.$$

Consequently, from Theorem 4.12 we obtain,

$$\|(S_1X - XS_o) - J\| \geq \|(PY - YP) - S\| \geq \|S\| = \|J\|.$$

$\square$

Proposition 4.18. *Let $S, S_o \in F(H)$. If $S_o \in [F(H)]^{-1}$ and $\|S\|\|S_o^{-1}\| \leq 1$, then $\|\delta_{S,S_o} + J\| \geq \|J\|$ for all $X \in F(H)$ and $J \in Ker\delta_{S,S_o}$.*

Proof. Let $J \in F(H)$ such that $SJ = JS_o$. Therefore, $SJS_o^{-1} = J$. But $\|S\|\|S_o^{-1}\| = 1$. It follows from [13] Theorem 2.1 that

$$\|SYS_o^{-1} - Y + J\| \geq \|J\|, \forall Y \in F(H).$$

If we set $X = YS_o^{-1}$ then we obtain,

$$\|(SX - XS_o) + J\| \geq \|J\| \text{ for all } X \in F(H).$$

But $\delta_{S,S_o}(X)$=$SX - XS_o$.
Hence, $\|\delta_{S,S_o}(X) + J\| \geq \|J\|$, for all $J \in Ker\delta_{S,S_o}$ and for all $X \in F(H)$. □

Remark 4.19. If $(J, |||.|||)$ is a norm ideal then the norm $|||.|||$ is unitarily invariant such that $|||SXP||| = |||T|||$ for all $T \in J$ and for all unitary operators.

Remark 4.20. Let $(J, |||.|||)$ be a norm ideal and $S, P \in L(H)$. If S is isometric and P contractive, then

$$|||\delta_{S,P}(X) + T||| \geq |||T||| \text{ for all } X \in J \text{ and for all } T \in Ker\delta_{S,P}.$$

Proposition 4.21. *Let $(J, |||.|||)$ be a norm ideal and $S \in F(H)$. Suppose that $f(S)$ is an operator that is subnormal and cyclic, where f is a function that is analytic and nonconstant on an open set containing $\sigma(S)$. Then*

$$|||\delta_S(X) + T||| \geq |||T||| \textit{ for all } X \in J \textit{ and for all } T \in \{S\} \cap J.$$

Proof. Let $T \in J$ such that $ST = TS$. This implies that $Tf(S) = f(S)T$ and $Sf(S) = f(S)S$. Since $f(S)$ is a cyclic subnormal operator, it follows from [93] that S and T are subnormal. But every subnormal operator is hyponormal [23]. Therefore, T is normal. Consequently, $ST = TS$ implies that $ST^* = T^*S$ by Putnam-Fuglede Theorem 3.3.4. Hence, $\overline{Ran(T)}$ and $Ker(T)^{\perp}$ reduces S and $S\mid_{\overline{R(T)}}$ and $S\mid_{Ker(T)^{\perp}}$ are normal operators. Let $T_o x = T_x$ for each $x \in Ker(T)$, it results that $\delta_{S,P}(T_o) = \delta_{S^*,P^*}(T_o) = 0$. Let $S = S_1 \oplus S_2$ with respect to $H = \overline{R(T)} \oplus \overline{R(T)}^{\perp}$ and $P = P_1 \oplus P_2$ with respect to $H = Ker(T)^{\perp} \oplus Ker(T)$. Then we can define S, T and X as follows
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$.
Then,

$$|||(SX - XS) + T||| = \left|\left|\left| \begin{pmatrix} S_1X_1 - X_1S_1 + T_1 & 0 \\ 0 & 0 \end{pmatrix} \right|\right|\right|.$$

This implies that

$$|||(SX - XS) + T||| \geq |||S_1X_1 - X_1S_1 + T_1||| \geq |||T_1|| = \|T|||.$$

Hence, $|||\delta_S(X) + T||| \geq |||\delta_{S_1}(X) + T_1||| \geq |||T_1||| = |||T|||$. □

Proposition 4.22. *Let $S, P \in F(H)$ such that the pair (S, P) possesses the PF property. Then, $|||\delta_{S,P} + T||| \geq |||T|||$ for all $X \in J$ and $T \in Ker\delta_{S,P}$.*

Proof. Let $T \in J$, since the pair S, P satisfies PF property. Then, $\overline{Ran(T)}$ decreases S and $Ker(T)^{\perp}$ decreases P and $S\mid_{\overline{Ran(T)}}$ and $P\mid_{Ker(T)^{\perp}}$ are normal operators. Let $T_o : Ker(T)^{\perp} \to \overline{Ran(T)}$ be defined by setting

$T_o x = T_x$ for each $x \in Ker(T)$, it follows that $\delta_{S,P}(T_o) = \delta_{S^*,P^*}(T_o) = 0$. Let $S = S_1 \oplus S_2$ in relation to $H = \overline{Ran(T)} \oplus \overline{Ran(T)}^{\perp}$ and $P = P_1 \oplus P_2$ with respect to $H = Ker(T)^{\perp} \oplus Ker(T)$. Let S, P,T and X have the following representation.
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$.
From Proposition 4.2 we have,

$$|||(SX - XP) + T||| = \left|\left|\left| \begin{pmatrix} (S_1X_1 - X_1P_1) + T_1 & 0 \\ 0 & 0 \end{pmatrix} \right|\right|\right|.$$

This means that

$$|||(SX - XP) + T||| \geq |||(S_1X_1 - X_1P_1) + T_1||| \geq |||T_1||| = |||T|||.$$

Hence, $|||\delta_{S,P}(X) + T||| \geq |||\delta_{S_1,P_1}(X) + T_1||| \geq |||T_1||| = |||T|||$. □

Proposition 4.23. *Let $S, P \in F(H)$ be normal operators such that $SP = PS$ and $S^*S + P^*P > 0$. For an elementary operator $E(X) = SXP - PXS$, $|||E(X) + J||| \geq |||J|||$ for all $J \in KerE$.*

Proof. Assume that $P^{-1} \in L(H)$, then from $SP = PS$ and $SJP = PJS$ we get, $SP^{-1}J = JP^{-1}S$. Therefore if theorem AK [81] is applied to the operators SP^{-1}, $P^{-1}S$ and J we get,

$$|||(SX - XS) + J||| \geq |||(SP^{-1}PXP - PXP^{-1}S) + J||| \geq |||J|||.$$

Consider now the case when P is injective i.e $KerP = 0$. Let $\sigma_n = \{\lambda \in \mathbb{C} : \lambda \leq \frac{1}{n}\}$ and let $E_P(\sigma_n)$ be the respective spectral projector. If we put $P_n = I - E_P(\sigma_n)$. The subspace P_nH decreases both S and P (since they commute and are normal). Therefore, in relation to the decomposition $H = (I - P_n)H \oplus P_n(H)$

$S = \begin{pmatrix} 0 & 0 \\ 0 & S_1^{(n)} \end{pmatrix}, P = \begin{pmatrix} 0 & 0 \\ 0 & P_1^{(n)} \end{pmatrix} J = \begin{pmatrix} J_{11}^{(n)} & J_{12}^{(n)} \\ J_{21}^{(n)} & J_{22}^{(n)} \end{pmatrix}$ and $X = \begin{pmatrix} X_{11}^{(n)} & X_{12}^{(n)} \\ X_{21}^{(n)} & X_{22}^{(n)} \end{pmatrix}.$

It can be seen that $P_1^{(n)}$ acting on $P_n(H)$ is invertible. It follows that

$$\begin{aligned} |||SXP - PXS + J||| &\geq |||P_n(SXP - PXS + J)P_n||| \\ &= |||S_1^{(n)}X_{22}^{(n)}P_1^{(n)} - P_1^{(n)}X_{22}^{(n)}S_1^{(n)} + J_{22}||| \\ &\geq |||J_{22}||| = |||P_nJP_n||| \end{aligned}$$

Therefore, we have $|||SXP - PXS + J||| \geq |||P_nJP_n|||$.

Applying Lemma 3 [81] we obtain $|||SXP - PXS + J||| \geq |||J|||$.

Now, we assume that $KerS \cap KerP = \{0\}$. Let S, P, J and X have the following representation in relation to the space decomposition $H = KerP \oplus H_o(H_o \ominus KerP)$.

$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, P = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}, J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$ and $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$

Operators S_1 and P_2 are injective and we have,

$(SXP - PXS) = \begin{pmatrix} 0 & S_1X_{12}P_2 \\ -P_2X_{21}S_1 & S_2X_{22}P_2 - P_2X_{22}S_2 \end{pmatrix}.$

Since $SJP = PJS = 0$, then $S_2J_{22}P_2 = P_2J_{22}S_2$ and $S_1J_{12}P_2 = P_2J_{21}S_1 =$

0 since S_1 and P_2 are injective and their ranges are dense. We have,

$$\begin{aligned}
|||SXP - PXS + J||| &= \left|\left|\left| \begin{pmatrix} 0 & S_1X_{12}P_2 \\ -P_2X_{21}S_1 & S_2X_{22}P_2 - P_2X_{22}S_2 \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \right|\right|\right| \\
&= \left|\left|\left| \begin{pmatrix} J_{11} & S_1X_{12}P_2 \\ -P_2X_{21}S_1 & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} \right|\right|\right|.
\end{aligned}$$

Since P_2 is injective, we have already shown that

$$|||S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22}||| \geq |||J_{22}|||$$

Applying Lemma GK in [81] and Lemma 2 in [81] we have

$$\begin{aligned}
|||S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22}||| &\geq \left|\left|\left| \begin{pmatrix} J_{11} & 0 \\ 0 & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} \right|\right|\right| \\
&\geq \left|\left|\left| \begin{pmatrix} J_{11} & 0 \\ 0 & J_{22} \end{pmatrix} \right|\right|\right| = |||J|||.
\end{aligned}$$

□

Theorem 4.24. *Let $S, P \in L(H)$ be normal operators such that $PS = SP$ and $E(X) = SXP - PXS$. If $J \in KerE$ then*

$$|||E(X) + J||| \geq 3^{-1}|||J||| \tag{4.3.1}$$

and

$$\|E(X) + J\|_p \geq 2^{|1-\frac{2}{P}|}\|J\|_p, \tag{4.3.2}$$

where $\|.\|_P$ is the C_P norm.

In particular, for the Hilbert Schmidt-norm we have

$$\|E(X)+J\|_2^2 \geq \|J\|_2^2 + \|E(X)\|_2^2. \tag{4.3.3}$$

Proof. Let S, P, J and X have the following representation with respect to the space decomposition $H = H_1 \oplus H_2$, where $H_1 = KerS \cap KerP$) and $H_2 = H \ominus H_1$
$S = \begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}$, $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$ and $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$.
We have that $KerS_2 \cap KerP_2 = \{0\}$ in H_2. Applying Lemma 1 in [81] and Proposition 4.23 we have,

$$\begin{aligned}
|||SXP - PXS + J||| &= \left|\left|\left| \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} \right|\right|\right| \\
&\geq |||S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22}||| \\
&\geq 2^{-1}\,|||S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22}||| \\
&\geq 2^{-1}\,|||S_2X_{22}P_2 - P_2X_{22}S_2||| \\
&\geq 2^{-1}\,|||SXP - PXS|||\,.
\end{aligned}$$

For us to prove Inequality 4.3.2 we begin with the initial similar inequalities and then we apply Lemma K in [81] twice and Proposition 4.23. For $1 \leq p \leq 2$ we have,

$$\begin{aligned}\|SXP - PXS + J\|_p^p &= \left\| \begin{pmatrix} 0 & 0 \\ 0 & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} + \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \right\|_p^p \\ &= \left\| \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} \end{pmatrix} \right\|_p^p \\ &\geq 2^{p-2}(\|J_{11}\|_p^p + \|J_{12}\|_p^p + \|J_{21}\|_p^p \\ &+ \|S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} + J_{22}\|_p^p) \\ &\geq 2^{p-2}(\|J_{11}\|_p^p + \|J_{12}\|_p^p + \|J_{21}\|_p^p + \|J_{22}\|_p^p) \\ &\geq 2^{p-2}\|J\|_p^p.\end{aligned}$$

and for $2 \leq p < \infty$ we have,

$$\begin{aligned}\|SXP - PXS + J\|_p^p &= \left\| \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} + J_{22} \end{pmatrix} \right\|_p^p \\ &\geq 2^{2-p}(\|J_{11}\|_p^p + \|J_{12}\|_p^p + \|J_{21}\|_p^p \\ &+ \|S_2X_{22}P_2 - P_2X_{22}S_2 + J_{22} + J_{22}\|_p^p) \\ &\geq 2^{2-p}(\|J_{11}\|_p^p + \|J_{12}\|_p^p + \|J_{21}\|_p^p + \|J_{22}\|_p^p) \\ &\geq 2^{2-p}\|J\|_p^p.\end{aligned}$$

Hence, $\|SXP - PXS + J\|_p^p \geq 2^{|1-\frac{2}{p}|}\|J\|_p^p$ and this proves Inequality 4.3.2 Now, if $p = 2$ Inequality 4.3.2 becomes $\|E(X) + J\|_2 \geq \|J\|_2$ and from Remark 1 in [81] this implies Inequality 4.3.3. □

Corollary 4.25. *Let $S, P \in L(H)$ be normal, then for every operator J satisfying $SJP = J$, $\|SXP - PXS + J\| \geq \|J\|$ for all $X \in L(H)$.*

Proof. Let $SJP = J$, then $SJ = JP^{-1}$. Since $SJP = J$ we have that $SJP = PJS$ which implies that $SP^{-1}J = JP^{-1}S$. Applying theorem AK [81] to the operators SP^{-1}, $P^{-1}S$ and J and from Proposition 4.23 we get

$$\|SXP - PXS + J\| = \|SP^{-1}PXP - PXPP^{-1}S + J\| = \|J\|.$$

Now, suppose P is not injective with respect to the decomposition $H = Ker(P)^{\perp} \cap KerP$. Using the condition $SJP = J$ we have,
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$, $P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}$, $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$ and $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$.

where S_1 is injective, from Proposition 4.23 it follows that

$$\begin{aligned} \|SXP - PXS + J\| &= \left\| \begin{pmatrix} S_1X_1P_1 - P_1X_1S_1 & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \right\| \\ &\geq \left\| \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \right\| \\ &\geq \|J\|. \end{aligned}$$

□

Corollary 4.26. *If the assumptions of Theorem 4.24 hold, then* $\overline{ranE} \cap KerE = \{0\}$ *where the closure can be taken in the more uniform norm. Hence* $E(E(X)) = 0$ *implies that* $E(X) = 0$.

Proof. If $S \in \overline{ranE} \cap KerE$, then $S = lim_{n\to\infty} E(x_n)$ and $E(S) = 0$.

From theorem 2.1 in [81] we have that

$$\|E(x_n) - S\| \geq c\|S\|.$$

Hence,

$$\|S - S\| \geq c\|S\|.$$

Therefore,

$$S = 0.$$

□

4.4 Birkhoff-James orthogonality for finite elementary operators

In this section we determine Birkhoff-James orthogonality for finite elementary operators. In [70] we have that for the examples elementary given in Section 1.2 (inner derivation, generalized derivation, basic elementary operator, Jordan elementary operator) the following implication hold for a general bounded linear operator S on a normed linear space Ω. i.e $Ran(S) \perp KerS \Rightarrow \overline{Ran(S)} \cap KerS = 0 \Rightarrow Ran(S) \cap KerS = 0$, where $\overline{Ran(S)}$ denotes the closure of the Range of S and $KerS$ denotes the Kernel of S and $Ran(S) \perp KerS$ means Range of S is orthogonal to the Kernel of S in the sense of Birkhoff.

Proposition 4.27. *Let $S \in L(H)$ be isometric, then $Ran\delta_S \perp Ker\delta_S$.*

Proof. From Proposition we know,

$$S^nX - XS^n = \sum_{i=0}^{n-i-1}(SX - XS)S^i \text{ for all } X \in L(H).$$

Therefore if $SJ = JS$ we have,

$$nJS^{n-1} = S^nX - XS^n - \sum_{i=0}^{n-i-1} S^{n-i-1}((SX - XS) - J)S^i \text{ for all } X \in L(H).$$

When both sides are divided by n and if we take norms we have,

$$\|JS^{n-1}\| \leq \frac{1}{n}\|S^nX + XS^n\| + \frac{1}{n}\sum_{i=0}^{n-i-1} \|S^{n-i-1}\|\|((SX - XS) - J)\|\|S^i\|.$$

Since S is isometric we have,

$$\|J\| \leq \frac{2}{n}\|X\| + \|((SX - XS) - J)\|.$$

Letting $n \to \infty$ we obtain,
$\|(SX - XS) - J\| \geq \|J\|$ and hence, $Ran\delta_S \perp Ker\delta_S$. □

Corollary 4.28. *Let $S, S_o \in L(H)$ be contractive such that $\delta_{S,S_o}(J) = 0$ for some $J \in L(H)$. Then*

$$\|\delta_{S,S_o} + J\| \geq \|J\| \textit{ for all } X \in L(H).$$

Proof. Given $J \in L(H)$ and from Proposition 4.27 we have,

$$nJS_o^{n-1} = S^nX - XS_o^n - \sum_{i=0}^{n-i-1} S^{n-i-1}((SX - XS_o) - J)S_o^i \text{ for all } X \in L(H).$$

If both sides are diveded by n and if norms are taken we have,
$$\|JS_o^{n-1}\| \leq \frac{1}{n}\|S^nX + XS_o^n\| + \frac{1}{n}\sum_{i=0}^{n-i-1} \|S^{n-i-1}\|\|((SX - XS_o) - J)\|\|S_o^i\|.$$

But S and S_o are contractive i.e $\|S^n\| \leq 1$ and $\|S_o^n\| \leq 1$. This implies that $\|S^{n-1}\| \leq 1$ and $\|S_o^{n-1}\| \leq 1$ and hence we have,

$$\|J\| \leq \tfrac{2}{n}\|X\| + \|((SX - XS_o) - J)\|.$$

Letting $n \to \infty$ we obtain,

$$\|(SX - XS_o) - J\| \geq \|J\|. \text{ Therefore, } Ran\delta_{S,S_o} \perp Ker\delta_{S,S_o}.$$

□

Lemma 4.29. *Let $S, P \in L(H)$, such that the pair (S, P) satisfies (PF) property, then $Ran\delta_{S,P} \perp Ker\delta_{S,P}$.*

Proof. Suppose $X \in Ker\delta_{S,P}$, then $SX - XP \in Ran\delta_{S,P} \cap Ker\delta_{S,P}$. For $J \in Ker\delta_{S,P}$, we have that the $\overline{Ran(J)}$ decreases S and $Ker(J)^{\perp}$ reduces P and $S \mid_{\overline{Ran(J)}}$ and $P \mid_{Ker(J)^{\perp}}$ are normal operators. Let S, P,J and X have the following representation in relation to the decompositions $H = H_1 = \overline{R(J)} \oplus \overline{R(J)}^{\perp}$, $H = H_2 = Ker(J)^{\perp} \oplus Ker(J)$.
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$.
From Proposition 4.8 we have,

$$\|(SX - XP) + J\| = \left\| \begin{pmatrix} (S_1X_1 - X_1P_1) + J_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|.$$

This implies that

$$\|(SX - XP) + J\| \geq \|(S_1X_1 - X_1P_1) + J_1\| \geq \|J_1\| = \|J\|.$$

Hence, $\|\delta_{S,P}(X) + J\| \geq \|\delta_{S_1,P_1}(X) + J_1\| \geq \|J_1\| = \|J\|$.

Therefore, $Ran\delta_{S,P} \cap Ker\delta_{S,P} = 0$. □

Remark 4.30. Let $S \in L(H)$ be quasihyponormal and T^* be injective hyponormal operator, if $ST = TS$ for some $X \in L(H)$. Then $S^*T = T^*S$, $RanJ$ reduces S, $KerJ^\perp$ reduces T and $S \mid_{\overline{Ran(J)}}$ and $T \mid_{Ker(J)^\perp}$ are unitarily equivalent normal operators.

Theorem 4.31. *Let $S \in L(H)$ be quasihyponormal and T^* be injective hyponormal operator in $L(H)$, then $Ran\delta_{S,T} \perp Ker\delta_{S,T}$.*

Proof. The pair (S,T) has the $PF_{(LH)}$ property by Remark 4.30. Let $J \in L(H)$ be such that $SJ = JT$. Let S, T, J and X have the following representation in relation to the decompositions $H = K = \overline{Ran(J)} \oplus \overline{Ran(J)}^\perp$, $H = L = Ker(J)^\perp \oplus Ker(J)$.

$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$.

where T_1 and S_1 are normal operators on K to L, then we have,

$$\|(SX - XT) + J\| = \left\| \begin{pmatrix} (S_1X_1 - X_1T_1) + J_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|.$$

Thus, from Lemma 4.29 it follows that

$$\|(SX - XT) + J\| \geq \|(S_1X_1 - X_1T_1) + J_1\| \geq \|J_1\| = \|J\|.$$

Hence, $Ran\delta_{S,T} \perp Ker\delta_{S,T}$. □

Let $E(X) = SXS_o - S_oXS$, then we have the following theorem.

Theorem 4.32. *Let $S,\ S_o \in L(H)$ be normal operators such that $SS_o = S_oS$. Then $\|(SXS_o - S_oXS) + J\|_p \geq \|J\|_p$, for all $X \in C_p$ and for all $J \in KerE \cap C_p$ $(1 \leq p < \infty)$.*

Proof. Taking the Hilbert space $H \oplus H$ and considering the operators.
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}, J = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$.
It follows that

$$\|(SXS_o - S_oXS) + J\|_p = \left\| \begin{pmatrix} (S_1X_1S_2 - S_2X_1S_1) + J_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|_p .$$

Thus, from Theorem 4.31 we have

$$\|(SXS_o - S_oXS) + J\|_p \geq \|(S_1X_1S_2 - S_2X_1S_1) + J_1\|_p \geq \|J_1\|_p = \|J\|_p.$$

Hence, $RanE \perp KerE$. □

Let $\varphi(X) = SXS_o - PXP_o$, then we have the following corollary.

Corollary 4.33. *Let $S,\ S_o,\ P,\ P_o \in L(H)$ be normal operators such that $SP = PS$ and $S_oP_o = P_oS_o$. Then $\|(SXS_o - PXP_o) + J\|_p \geq \|J\|_p$, for all $X \in C_p$ and for all $J \in Ker\varphi \cap C_p$ $(1 \leq p < \infty)$.*

Proof. On $H \oplus H$ consider the operators S, S_o, P, P_o, J and X defined by
$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, S_o = \begin{pmatrix} S_2 & 0 \\ 0 & 0 \end{pmatrix}, P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}, P_o = \begin{pmatrix} P_2 & 0 \\ 0 & 0 \end{pmatrix},$
$J = \begin{pmatrix} J_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}$.
It follows that

$$\|(SXS_o - PXP_o) + J\|_p = \left\| \begin{pmatrix} (S_1X_1S_2 - P_1X_1P_2) + J_1 & 0 \\ 0 & 0 \end{pmatrix} \right\|_p .$$

Thus, from Theorem 4.32 we have

$$\|(SXS_o - PXP_o) + J\|_p \geq \|(S_1X_1S_2 - P_1X_1P_2\|_p \geq \|J_1\|_p = \|J\|_p.$$

Hence, $Ran\varphi \perp Ker\varphi$. □

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this chapter, conclusions are drawn and recommendation made based on the objectives of the study and the results obtained.

5.2 Conclusion

Results for characterization of finiteness of elementary operators has been obtained in this study. The first objective of this study has been to characterize finiteness of elementary operators. Given that an operator S on a normed space Ω is finite if $\|(SX-XS)-I\| \geq 1$, we investigated finiteness of elementary operators(inner derivation, generalized derivation, basic elementary operator, Jordan elementary operator) and those are operators of the form $E(X) = \sum_{i=1}^{n} A_i X B_i$. For those operators we defined their finiteness by $E(X) = \|\sum_{i=1}^{n} A_i X B_i - I\| \geq 1$. In this case we proved that elementary operators are finite. Hence, the main results of finiteness of elementary operators are in line with the stated objective.

In objective two, we presented some orthogonality conditions for finite elementary operators. In this case, we proved that for the operators $S, P \in L(H)$ the range of $\delta_{S,P}$ is orthogonal to its kernel if the pair of operators are normal and they commute, we extended this results to unitarily invariant norms, where we showed that for the operators $S, P \in L(H)$ the range of $\delta_{S,P}$ is orthogonal to its kernel if the pair of operators satisfy Putnam Fulgede Property. We also had other conditions such as, for operators $S, S_o \in F(H)$ the range of δ_{S,S_o} is orthogonal to its kernel if S is invertible and $\|S\|\|S_o\| \leq 1$. Hence, the main results of orthogonality conditions for finite elementary operators obtained are in conformity with the stated objective.

Finally, in objective three, we characterized Birkhoff-James orthogonality for finite elementary operators, we showed that for the inequality $\|(SX - XP) - J\| \geq \|J\|$ means that the Range of $\delta_{S,P}$ is orthogonal to the kernel of $\delta_{S,P}$ in the sense of Birkhoff. Hence, the main results of Birkhoff-James orthogonality for finite elementary operators correspond to the stated objective.

Therefore, the main results of finiteness of elementary operators, orthogonality conditions for finite elementary operators and Birkhoff-James for finite elementary operators obtained in this study are in line with the stated objectives.

5.3 Recommendations

The results obtained are specific to finiteness of elementary operators, orthogonality conditions for finite elementary operators and Birkhoff-James for finite elementary operators in complex normed spaces.

In the first objective, we showed that $\mathfrak{C}_n(S, S_o) \subset F(H)$ through Remark 4.13 together with their corollaries. Hence, it is interesting to pose an open problem that follows analogously from remark 4.13 as below.
Open problem. Is $F(H) \subset \mathfrak{C}_n(S, S_o)$ in a general Banach space setting?

For objective two, we showed that the range of finite elementary operators is orthogonal to its kernel if the operators satisfy Putnam Fuglede property. Therefore, the open problem here is to find nonnormal operators satisfying Putnam Fuglede property and consequently the range kernel orthogonality results can be obtained.

Lastly, the results in objective three, can be extended to general Banach space setting and Birkhoff-James orthogonality can be determined for nonnormal operators.

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