LIE SYMMETRY SOLUTION OF THIRD ORDER FIRST DEGREE NONLINEAR WAVE EQUATION OF FOURTH DEGREE IN SECOND DERIVATIVE

BY

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DEDICATION

To my beloved wife, Prisca Anupi,

tiny Matete Lenzclif,

small Aluala Arnold and

young Toko Frank

for your patience, support and prayers .

“Were kabalinde koo, kabamete kamaani!”

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ABSTRACT

In this study the method of Lie symmetry was used to determine the solution to a third order first degree nonlinear ordinary differential equation (ODE) fourth degree in second derivative that arise in waves of systems like water in shallow oceans. Many third order nonlinear ordinary differential equations (ODEs) have been developed using numerical methods like the finite difference but their solutions are just approximations within known boundary conditions or restrictions. To address such limitations, analytical Lie symmetry method which provides group invariant solutions was applied. This method does not depend on initial boundary values and gives exact solutions to problems. It has been shown what Lie symmetry analysis entails by reviewing some relevant nonlinear ordinary differential equations which have admitted it. The solution to nonlinear ordinary differential equation of the general form:

$$G(x, y, y', y'', y''') = 0$$

that has not been developed by other earlier researchers has been worked out sequentially. A comprehensive Lie symmetry analysis carried out on this nonlinear ordinary differential equation included Lie groups, Lie symmetry generators, prolongations, invariant transformations, integrating factors and order reduction. The most significant Lie group theory application used was the order reduction of the nonlinear ODE from a third order to a first order which is easily solvable by other known simple methods. The objectives were to develop and determine both mathematical solution and general solution to a third order first degree nonlinear ODE of fourth degree in the second derivative, a special case of wave equation whose form was

$$y''' - y'\left(\frac{y''}{y}\right)^4 = 0$$

using Lie symmetry method. Its solution is the source of knowledge and basis for further future research.

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CHAPTER 1

INTRODUCTION

1.1 Background of the Study

A differential equation is an equation in which at least one term contains any differential coefficients such as $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ and $\frac{d^5y}{dx^5}$ whose solution is an equation relating x and y which contains no differential coefficients. It is an equation which involves a function, its derivative and the independent variables. If it has only one independent variable, then it is called an ordinary differential equation. This means that it is a relationship between an independent variable x, a dependent variable y and one or more derivatives of y with respect to x. For example:

$$x + 2\frac{dy}{dx} = 3y \tag{1.1}$$

$$x^2\frac{dy}{dx} = ysin(x) \tag{1.2}$$

$$xy\frac{d^2y}{dx^2} + y\frac{dy}{dx} + e^{3x} = 0 \tag{1.3}$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = x^2 \tag{1.4}$$

Any differential equation represents a dynamic relationship, that is, quantities that change, say x and y and are thus frequently occurring in scientific and engineering problems. A differential equation is either a linear equation or a nonlinear equation. A differential equation is said to be linear if it is linear in its dependent variable, for example:

$$y^{''}sin(x) + x^2y = 0 \tag{1.5}$$

A linear equation in a single variable (unknown) involves powers of a variable no higher than the first. A linear equation is also referred to as a single equation. A differential equation given by

$$F(x, y, y^{'}, ..., y^{n}) = 0 \tag{1.6}$$

is linear if the function F is a linear function of variables $y, y^{'}, ..., y^{n}$. Thus the general linear differential equation of order n may be written as:

$$b_0(x)\frac{d^n y}{dx^n} + b_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + b_{n-1}(x)\frac{dy}{dx} + b_n(x)y = R(x) \tag{1.7}$$

where R is a function of x and $b_0, b_1, ..., b_n$ are known constants. The term linear refers to the fact that each expression in the differential equation is of degree one or zero in the variables: $y, y^{'}, ..., y^{n}$. If $R(x) = 0$, then the differential equation is said to be linear and homogeneous as in

$$y^4 - y = 0 \tag{1.8}$$

which is homogeneous of order four with constant coefficients. A function $f(x, y)$ is said to be homogeneous of degree n if on replacing x and y by Δx and Δy , where Δ is a parameter, we have:

$$f(x, y) = x^4 - x^3 y \tag{1.9}$$

is homogeneous of degree four since:

$$f(\Delta x, \Delta y) = (\Delta x)^4 - (\Delta x)^3 \Delta y$$

$$= \Delta^4 x^4 - \Delta^4 x^4 y$$

$$= \Delta^4 (x^4 - x^3 y)$$

$$= \Delta^4 f(x, y) \tag{1.10}$$

otherwise if $R(x) \neq 0$, then the differential equation is said to be linear and inhomogeneous as in

$$xy^{'} - 2y = x^3 \tag{1.11}$$

is inhomogeneous of order one with variable coefficients. This implies that for a linear differential equation, none of the differential coefficients is raised to a power other than one, for example,

$$\frac{dy}{dx} = 3x \tag{1.12}$$

On the other hand, a nonlinear differential equation is one whose differential coefficients are raised to a power greater than one, for instance,

$$y^{'} + y^2 x = 0 \tag{1.13}$$

$$x^2 + 2\left(\frac{dy}{dx}\right)^2 + 4y = 0 \tag{1.14}$$

A differential equation can be homogeneous or inhomogeneous. A differential equation is homogeneous if it has no terms that are functions of the independent variable alone. That is, a homogeneous differential equation is in which every term is of the same degree, for instance,

$$y^{''} + yx + y = 0 \tag{1.15}$$

$$a^2 - 3ab - 40b^2 = 0 \tag{1.16}$$

$$a^3 - 3a^2 b + 4b^3 = 0 \tag{1.17}$$

$$6x^3 + 7x^2 y - 7xy^2 - 6y^3 = 0 \tag{1.18}$$

$$4x^4 - 37x^2 y^2 + 9y^4 = 0 \tag{1.19}$$

Differential equations used in coordinate geometry are homogeneous, for example, the parabola $y^2 = 4ac$, an ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the rectangular hyperbola $xy = c^2$. The advantage of using the homogeneous differential equations is that the equations derived from them are also homogeneous and this gives a method of detecting the slips. Some useful mathematical identities are provided inhomogeneous forms:

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \tag{1.20}$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 \tag{1.21}$$

An inhomogeneous differential equation is one in which there are terms that are functions of the independent variables alone. A differential equation whose terms are not of the same degree is called a inhomogeneous equation, for example,

$$y' + y + x^3 = 0 \tag{1.22}$$

$$x^2 + y^2 + 2x + 2y = 1 \tag{1.23}$$

$$x^3 + y^3z + z^3x = 0 \tag{1.24}$$

There are two categories of differential equations namely ordinary differential equations (ODEs) and partial differential equations (PDEs). Any relation between the variables x, y and the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, ...$ is called an ordinary differential equation (ODE). The term ordinary distinguishes it from a partial differential equation which involves the partial derivatives. This study considered a nonlinear ODE in its work. A differential equation with one independent variable present is called an ODE, for example,

$$y''(x) + y(x) = 0 \tag{1.25}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \tag{1.26}$$

$$\frac{dy}{dx} = cos(x) \tag{1.27}$$

$$y''' + 4y'' + 2y' - 6y = 0 \tag{1.28}$$

whereas that with more than one independent variable is known as a PDE, for example,

$$\frac{y^2z}{x}p + xzq = y^2 \tag{1.29}$$

$$p + 3q = 5z + tan(y - 3x) \tag{1.30}$$

$$(yz + xyz)dz + (xz + xyz)dy + (xy + xyz)dz = 0 \tag{1.31}$$

ODEs yield linearly dependent and linearly independent solutions. A set of functions $f_1, f_2, ..., f_n$ are said to be linearly dependent if there exist constants $C_1, C_2, ..., C_n$ not all zero such that

$$C_1f_1(x) + C_2f_2(x) + ... + C_nf_n(x) = 0 \tag{1.32}$$

$\forall x \epsilon f = [a, b]$ otherwise they are linearly independent. For instance, consider a second order linear differential equation:

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{1.33}$$

$a_0 \neq 0$, whose two solutions are $y_1(x)$ and $y_2(x)$ are said to be linearly dependent if there exist two constants C_1 and C_2 which are both not zero such that

$$C_1y_1(x) + C_2y_2(x) = 0 \tag{1.34}$$

Conversely, the two solutions $y_1(x)$ and $y_2(x)$ are linearly independent if they are not linearly dependent, that is, if

$$C_1y_1(x) + C_2y_2(x) = 0 \tag{1.35}$$

$\Rightarrow C_1 = 0$ and $C_2 = 0$. Further, if the Wronskian is zero then the two solutions are linearly dependent otherwise they are linearly independent. The Wronskian is the criteria for finding out whether the solutions of a linear differential equation are linearly dependent or not.

In mathematics, solving differential equations is the most significant driving force behind its history. A lot of literature is available about the differential equations and yet it is until recently that the group theory has been applied. The symmetry group of nonlinear ODEs is a group of transformations of independent and dependent variables that leave all solutions invariant. This symmetry group generates new solutions to a given nonlinear ODE which can be used to reduce its highest order to a first order. Solving of nonlinear ODEs after the introduction of derivatives and integrals was a major achievement. After this discovery, many methods involving differentiations and integrations were developed (Erdmann, et. al, 2006). When trying to find solutions to nonlinear ordinary differential equations, methods employed often terminate such that the trials are left without any show whether or not there is a solution. This led to the desire of developing a method that could be used in solving of the largest forms of nonlinear ordinary differential equations.

According to Yaglom, (1988) Norwegian mathematician Sophus Lie put forward many of the fundamental ideas behind symmetry methods. Late nineteenth century, Sophus Lie introduced the notion of Lie group to study the solutions of ODEs. The techniques of integration got extended and unified when he managed to reduce by one the order of point transformations under one - parameter Lie group of point transformations that is invariant. Lie embarked on developing these continuous groups that are now used in many career based sciences. Lie point symmetry of a system is a local group of transformations that maps every solution of the system to another solution of the same system. Rotations, scalings and translations are simple examples of Lie groups (Olver, 1993).

The significance of the outcomes for Lies theory comes from the symmetries of any differential equation found from its determining system of linear homogeneous partial differential equations (PDEs). By means of Loewys and Janets results for nonlinear ODEs may be decomposed in a number of basic problems whose solution can be designed. In this approach, the symmetries of a nonlinear ODE serve the main purpose of identifying its equivalence class or classes, for which a canonical form is known (Riley, et. al, 2008).

There are three stages in the scheme for solving second - order ODEs: First find the group type of symmetry. This is most efficiently achieved through Janet base coefficients for the system finding. Second, a canonical form of the transformed equation should correspond to its type symmetry and third, get the solution of the equation given by solving its canonical form.

According to Aminer, (2015) Sophus Lie came upon this kind of situation when he started tackling this problem. He recognized the transformation properties of a nonlinear ODE under certain groups of continuous transformations as being fundamental in analyzing its solution (Mehmet, 2004). It shows how derivatives of the dependent variable y with respect to the independent variable x relate.

The symmetry methods for solving differential equations were initially developed by Sophus Lie (1988). He introduced the notion of continuous groups known as Lie groups or symmetry groups for the applications of the differential equations and was based on the system of invariance under the Lie group of transformations. Therefore a symmetry group is a group of transformations that map any solution of the system onto another solution of the same system. Thus Lie group analysis is a mathematical theory which synthesizes the symmetries of different nonlinear ODEs. The nonlinear ODE is thus a function of x which is written as $y(x)$. It has a closed solution if $y(x)$ can be expressed in terms of the standard elementary functions like exp (x), ln (x), cos (x), sin (x), tan (x) (Kamke, 1967). Lie among other things, came up with a classification of differential equations in the terms of their symmetry groups, hence identifying the set of differential equations that could be integrated or reduced to a lower order equations by means of theoretical group arguments which are then simpler to solve. The basic idea of Lie was to find all Lie groups of the given nonlinear ODE so that any solution to this nonlinear ODE is transformed into another solution using the coordinate transformations of the respective Lie groups. This implies that all the Lie groups with respect to which the set of solutions of the nonlinear ODE is invariant and the solutions which result from this procedure are called Lie symmetry solutions. Lie groups represent a subject in which the algebraic groups and topological structures are both interlinked by the condition of continuity which involves the operation of group multiplication. Mathematical models of real life phenomena are formulated in the form of differential equations. The general theory of differential equations is one of the most essential applications of Lie group theory. One of the main problems of the group analysis of differential equations is to study the action of the group admitted by the given equation in a set of solutions to this equation. The action of the admitted group introduces into the set of solutions an algebraic structure which may be used to achieve the goals. It involves a description of the general properties of all the members of the family of solutions which are easier than the general solution.

The order of differential equation is the order of the highest order derivative present in the equation, for example,

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 + y = 0 \tag{1.36}$$

is a differential equation of order two and it is called a second order equation. Other examples are:

$$y^{'} = y^3 x \tag{1.37}$$

$$y^{''} + 2xy^{'} + 3y = x^2 \tag{1.38}$$

$$y^{'''} = yy^{''} \tag{1.39}$$

which are first, second and third order respectively. In general, the differential equation

$$F(x, y, y^{'}, ..., y^{(n)}) = 0 \tag{1.40}$$

is called as an n^{th} order. A first order ordinary differential equation is an equation that involves at most the first derivative of an unknown function. If y, the unknown function, is a function of x, then write the first order differential equation as

$$\frac{dy}{dx} = g(x, y) \tag{1.41}$$

where $g(x, y)$ is a given function of the two variables x and y. The highest derivative contained in a nonlinear ODE is its order, that is, the order of a differential equation is determined by the highest differential coefficient present. It is that of the highest derivative occurring in it. For example,

$$x\frac{dy}{dx} - y^2 = 0 \tag{1.42}$$

is called first order

$$xy\frac{d^2y}{dx^2} - y^2 sin(x) = 0 \tag{1.43}$$

is second order

$$\frac{d^3y}{dx^3} - y\frac{dy}{dx} + e^{4x} = 0 \tag{1.44}$$

is third order and so on (Iserles, et. al, 2000).

In mathematics, they can occur when arbitrary constants are eliminated from a given function. A function with one arbitrary constant gives a first order equation. A function with two arbitrary constants gives a second order equation. It can be generalized that an n^{th} order differential equation is derived from a function having n arbitrary constants. Mathematicians have solved nonlinear ordinary differential equations of orders higher than one and eventually obtained their solutions. Their solutions provided answers to nonlinear ODEs and also established a new field called the theory of groups, the basis of modern algebra (Schwarz, 1988).

Nonlinear ODEs are further grouped according to degree. The degree of a differential equation is the highest power of the derivative in the given equation, for example,

$$y^{'} - 2y^2 = cos(x) \tag{1.45}$$

$$y^3 + (y^{'})^2 = 0 \tag{1.46}$$

$$(y^{''})^3 + 2xsin(y) = e^x \tag{1.47}$$

which are first, second and third degree respectively. After a nonlinear ODE has been rationalized, the highest order derivative is raised to an index called the degree, this is to say that the degree of a differential equation is that to which the derivative of the highest order is raised when the equation is expressed in a rational integral form, hence the nonlinear ODE:

$$y''' - y'\left(\frac{y''}{y}\right)^4 = 0 \tag{1.48}$$

is a third order first degree nonlinear ordinary differential equation of fourth degree in the second derivative. Differential equations may be formed in practice terms from a consideration of the physical problems to which they refer. To solve a differential equation, the function for which the equation is true has to be developed. This means that the equation has to be manipulated so as to eliminate all the derivatives and leave a relationship between y and x. Many practical problems in engineering give rise to second and third order differential equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \tag{1.49}$$

$$a\frac{d^3y}{dx^3} + b\frac{d^2y}{dx^2} + c\frac{dy}{dx} + dy = f(x) \tag{1.50}$$

respectively, where a, b, c and d are constant coefficients and $f(x)$ is a given function of x. This nonlinear ODE is commonly used in many physical applications especially in engineering field and its very complex to be solved analytically. Differential equations have many ingenious but limited methods for obtaining exact solutions and they have a feature in common, namely they exploit their symmetries which lead to exact solutions (Dresner, 1999).

1.2 Statement of the Problem

Consider a nonlinear ODE of the form:

$y''' - y'\left(\frac{y''}{y}\right)^4 = 0$

whose general solution has been worked out analytically using the method of Lie symmetry or numerically using the method of finite difference or finite elements where the convergence of the numerical schemes wholly depend on the given initial boundary values. The adjoint-symmetries for the nonlinear ODE are obtained but the variational symmetries which are not invariant and the true symmetries which represent invariance under transformations of the nonlinear ODE are not known. The variational symmetries take up new variables as invariance under a continuous transformation that yield differential invariants whereas true symmetries lead to infinitesimal generators and they are geometric transformations such as translations, reflections or rotations. Therefore to be able to get the solution to the given nonlinear ODE, there is a great need to find all the Lie groups admitted and all the symmetries of the wave equation. It is due to this that this study has attempted to determine the solution to a special case of wave equation (1.48) which has been expressed as

$$y^{'''} - y^{'}(y)^{-4}(y^{''})^4 = 0 \tag{1.51}$$

using analytic method of Lie symmetry. The solution by this method neither depends on initial or boundary values nor is it an approximation to the exact solution.

1.3 Objectives of the Study

The objectives of this study were:

(i) To develop a mathematical solution to wave equation of third order first degree nonlinear ordinary differential equation of fourth degree in second derivative (1.48)

(ii) To determine a general solution using Lie symmetry analysis.

1.4 Significance of the Study

The solution to the wave equation (1.48) using Lie symmetry method shall become an extension of the analytic methods for solving third order first degree nonlinear ordinary differential equation of fourth degree in second derivative and other similar nonlinear ordinary differential equations in future. This is a boost the existing knowledge in solving mathematical problems using Lie symmetry analysis.

1.5 Scope of the Study

The study analyzed and generated a solution to a third order first degree nonlinear ordinary differential equation of fourth degree in the second derivative which is a special form of a wave equation (1.48). The method employed was Lie symmetry which developed and applied Lie groups, Lie algebras, infinitesimal transformations, prolongations, invariance under transformations, variational symmetries, Lie point symmetries, integrating factor and reduction of order.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This study looked at some of the works carried out in the same area of study which enabled the objectives to be achieved. This was done through borrowing what was relevant to the work, criticizing areas that do not add value to the work and extension of facts that are genuinely paramount to the research. The most recent works were studied, that is, the year 2015 and as early as the year 1918, which were relevant to this study area of interest.

2.2 Review of Solution to Nonlinear Wave Equation of Third Order

Opiyo,(2015) used the method of Lie symmetries to solve a third order first degree non-linear ordinary differential equation of cubic in the second order derivative and got a solution. The equation was of the form:

$$y''' = y\left(\frac{y''}{y'}\right)^3 \tag{2.1}$$

whose solution is:

$$V = \frac{1}{A(U')^4}\int U(U'')^3(U')^{-2}dU \tag{2.2}$$

He used Lie symmetry group invariant method where he applied Lie groups of transformations, Lie algebras, infinitesimal transformations, invariance under transformation, symmetry, Lie's integrating factor, method of canonical variables, Lie point symmetries and reduction of order. In this work infinitesimal transformations were used and extended this to degree four instead of degree three, symmetry, invariance under transformation, Lie's integrating factor and order reduction as applied in his research.The adjoint symmetries and method of canonical variables were not used in this work. The wave equation of study was actually an extension of his wave equation of research where the point of difference was the nonlinear term and the degree used. There is variation in the both solutions obtained and went a further step and attempted to come up with a general solution that holds true to all wave equations similar to the equation of study.

Aminer,(2014) applied Lie symmetry analysis in solving a fourth order nonlinear wave equation, a special type of nonlinear ODE. The form of the equation was:

$$(yy'(y(y')^{-1})'')' = 0 \tag{2.3}$$

and its solution is:

$$V = \frac{1}{U^3 + e^{2U'-1}} \int (U^3 + e^{2U'-1})(4U^{-1}U''^2U'^{-4} - 4U''^3U'^{-6} - U^{-2}U'^{-2}U'')dU \tag{2.4}$$

He developed Lie groups of transformations, Lie algebras,infinitesimal transformations, invariance under transformation, symmetry, Lie's integrating factor, method of canonical variables, Lie point symmetries, increasing of order and reduction of order. He used the symmetry transformations, symmetry reductions and global symmetry transformations to come up with all the solutions corresponding to each Lie group admitted by the special wave equation (2.3). The study applied the processes of finding infinitesimal transformations, Lie algebras, Lie groups of transformations, invariance under transformations, integrating factors and reduction of order during this research work.

Yulia, (2008) solved the equivalence problem of the third order ordinary differential equation which was quadratic in the second order derivative without a higher degree. For this group of differential equations the differential invariants of the group of the point equivalence transformations and the invariant differentiation operators were constructed. By using these results the differential invariants of thirteen Chazy equations were calculated. They provided examples of finding equivalent equations by the use of their invariants. Some new examples of the linearised equations by a local transformation were achieved. These are Schwarzian and Chazy equations. He employed Lie groups, infinitesimal generators, transformation maps and group invariants. All the above basic concepts were found to be applicable in this study. The only difference was in the wave equations studied where by it tackled an equation of both higher order and higher degree.

Leach, *et.* al,(2007) showed using group symmetries that the behaviour of a relativistic star is described by the following system of nonlinear ODEs:

$$\frac{1}{\gamma^2}[\gamma(1-e^{-2\lambda})]' = \rho \tag{2.5}$$

$$-\frac{1}{\gamma^2}(1-e^{-2\lambda}) + \frac{2v'}{\gamma}e^{-2\lambda} = p \tag{2.6}$$

$$e^{-2\lambda}(v'' + v'^2 + \frac{v'}{\gamma} - v'\lambda' - \frac{\lambda'}{\gamma}) = p \tag{2.7}$$

in a spherically symmetric space - time which is static. Here, primes denote differentiation with respect to the radial coordinate γ . The functions $v = v(\gamma)$ and $\lambda = \lambda(\gamma)$ represent the gravitational potential; $\rho = \rho(\gamma)$ and $p = p(\gamma)$ are the energy density and pressure respectively. Kweyama, (2005) showed that (2.5) - (2.7) may be represented in a number of equivalent forms to make the integration easier. It is convenient to use the transformation equations:

$$x = C\gamma^2 \tag{2.8}$$

$$Z(x) = e^{-2\lambda(\gamma)} \tag{2.9}$$

$$A^2y^2(x) = e^{2v(\gamma)} \tag{2.10}$$

Under the transformation (2.8) - (2.10), the Einstein field equations (2.5) - (2.7) take the form:

$$\frac{1-Z}{x} - 2\frac{dZ}{dx} = \frac{\rho}{C} \tag{2.11}$$

$$4Z\frac{1}{y}\frac{dy}{dx} + \frac{Z-1}{x} = \frac{p}{C} \tag{2.12}$$

$$4Zx^2\frac{d^2y}{dx^2} + 2x^2\frac{dZ}{dx}\frac{dy}{dx} + (x\frac{dZ}{dx} - Z + 1)y = 0 \tag{2.13}$$

and obtained the following solution:

$$
\begin{aligned}
& 2x^2y^3y''' + 2x^3y^2y'y''' - xy^2y'^2 + 4x^2yy'^3 + 2x^3yy'^4 \\
+ \quad & 5xy^3y'' - 2x^2y^2y'y'' + 2x^3yy'^2y'' - 6x^3y^2y''^2 = 0 \qquad (2.14)
\end{aligned}
$$

The basic concepts were the same to the ones used but origin of the wave equations differ. Leach looked at a wave equation of a star that is not in motion, that is, a stationary wave whereas it studied waves produced by movement of water masses in shallow oceans. Our solution was much simpler and shorter unlike Leach's solution that appears lengthy, tedious and uncondensed. It points out high chance of making further errors during the manipulation process.

Kweyama, (2005) researched on the role of Lie symmetries in generating solutions to differential equations that arise in particular physical systems. He looked at a nonlinear ODE arising from the field equations in the early universe cosmological models of the form:

$$2HH'' + 6H^2H' - H'^2 + aH^2 = b \qquad (2.15)$$

where $H = H(t)$ and got a quadratic equation of the form:

$$p^2 + 2pq - 1 = 0 \qquad (2.16)$$

which is a simple original second order ODE, where p and q are invariants. Systems of nonlinear differential equations appear in modeling physical phenomena arising in relativistic astrophysics in a similar manner they occur in water waves of shallow oceans. Kweyama used the following concepts in his study: Lie groups, Lie algebras, infinitesimal transformation, invariance under transformation, symmetry, Lie point symmetries, reduction of order, increasing of order, nonlocal symmetries and transformation of symmetries. In this study, all the concepts apart from increasing the order and nonlocal symmetries were applied. The integrating factors were used which did not feature in his work.

Oduor, (2005) used the method of symmetry to solve a generalized Burgers equation which is a nonlinear PDE found in wave theory. The form of the equation was:

$$u_t + uu_x = \lambda u_{xx} \tag{2.17}$$

and found its generalized global solution with no restriction to λ . He developed Lie groups, infinitesimal transformations, prolongations, Lie algebras, infinitesimal generators and the invariant transformations. Likewise infinitesimal transformations, generators, prolongations, Lie algebras, Lie groups and invariant transformations in this work were applied but the outstanding difference was that he tackled a PDE as opposed to ODE.

Mehmet, (2004) worked on the fourth order generalized Burgers equation by using Lie symmetry method. He confined himself to the application of Lie point symmetries, an application to fourth order. By using the computer programs under the computer package of mathematica, they found a three dimensional solvable Lie algebra of the point symmetries of the generalized Burgers equation four. They obtained the similarity reductions of these symmetries. The same Lie symmetry analysis was used when working on this wave equation. Lie algebras, reduction of order and point symmetries were applied in the research study.

Bluman and Anco, (2002) worked on how to find all the integrating factors and the corresponding first integrals for any system of ordinary differential equations. These integrating factors were shown to be all the solutions of both the adjoint system of the linearised system and a system which presents an extra adjoint invariance condition of the ordinary differential equations. They put forward an explicit construction formula to find the resulting first integrals in terms of integrating factors and the methods for getting these factors. More specific, the utilization of known first integrals and symmetries to find new integrating factors was advanced. The knowledge of group symmetries and integrating factors has been widely used in this study.

Omolo, (1997) used the method of Lie symmetry analysis of differential equations to solve a nonlinear PDE. He applied Lie groups, infinitesimal generators, prolongation, Lie algebras, symmetry and invariance under transformations. He employed a stability approach to exact solutions of the nonlinear PDE provided by symmetry groups. This study used the same concepts and yet the wave equation was a nonlinear ODE. He showed that the assumptions made while solving for the infinitesimals were not necessary because the conditions present themselves in a natural way.

Nucci, (1997) revisited solving of differential equations using the symmetries. He reviewed the role of symmetries in the solving of differential equations. The application of the classical Lie point symmetries in solving problems in meteorology, draining of fluid and epidemiology of AIDS also saw the use of non - classical symmetries. He showed that the iterations of non-classical symmetries method give new equations which are nonlinear and they inherit the Lie symmetry algebra of the given differential equation. Their differential invariants yield new solutions to the initial differential equation. It is this area of differential invariants that was borrowed and led to the required general solution of this wave equation.

Olver, (1993)used Lie symmetry analysis to come up with an existence theorem showing that if an n^{th} order ordinary differential equation admits $r-parameter$ Lie solvable group of transformations then it is the general solution of an $(n-r)^{th}$ order ordinary differential equation. He employed Lie group, Lie algebras, infinitesimal generators, group invariants and prolongations which have been applied to the wave equation that has been determined. He established a useful theorem applied to wave equations but in this case the theorem was not used because group invariants yielded the solution by applying simple known calculus methods.

Abraham-Shrauner, (1993) determined second order Lie symmetries of nonlinear ODEs and obtained a solution. He applied Lie groups, group invariants and order reduction to get his solution. The same basic concepts of Lie symmetry were applied to this equation of study. This wave equation was of a higher order than his, that is, order three against order two.

Schwarz, (1988) used the method of Lie symmetries to solve a second order differential equation and obtained a solution successfully. He worked out its Lie groups, infinitesimal generators, Lie algebras, prolongation and differential invariants which yielded the solution. In this equation of study, similar concepts were applied and got its general solution. Bluman and Cole, (1974) came up with a generalized Lie method known as the non - classical method of the group invariant solutions which had earlier been generalized by Olver and Rosenau, (1987). In this analysis, one replaces the conditions for the invariance of the given system of the differential equations by the weaker conditions for the invariance of the combined system consisting of the initial differential equations along with the equations requiring the group invariance of the solutions. By this device, a much wider class of groups is potentially available and hence there is the possibility of further kinds of explicit solutions being found by the same reduction techniques.

Spiegel, (1958) determined successfully the solutions of second order third degree in first order nonlinear ordinary differential equations using the method of symmetries. The basic concepts employed were Lie groups of transformations, Lie algebras, infinitesimal transformations, invariance under transformation, Lies integrating factor, Lie point symmetries and order reduction. This study borrowed similar concepts in tackling its wave equation. This study developed a higher equation in terms of order and degree, that is, order three and degree four.

Bianchi, (1918) used order reduction to solvable groups of Lie in mathematical systems which led him from higher order to lower order differential equations. He employed Lie groups, Lie algebras, infinitesimal transformations, differential invariants, integrating factor and reduction of order to achieve that. The lower equation was easier to solve by using other available methods, for example, quadratic equation methods and calculus.

CHAPTER 3

MATERIALS AND METHODOLOGY

3.1 Introduction

In order to determine the solution to equation (1.48) the following mathematical concepts will be applied: Lie groups, infinitesimal generators, Lie algebras, invariance under transformations, reduction of order, prolongations and integrating factors. In the first step Lie groups were developed which enabled to generate the infinitesimal generators. The application of prolongations onto the given equation resulted into non-zero Lie brackets that laid the foundation for getting differential invariants. These invariants were then used to reduce a third order to the first order which is easily solvable by other simpler known methods, for example integration. The use of integrating factors yielded both mathematical and general forms of the solution.

3.2 Lie Groups of Transformations

Definition 1: [A Group]

A group K is a non-empty set of elements with a law of composition Ω defined between the elements satisfying the following conditions (Olver, 1993):

(i) **Closure Property:** If x and y are elements of K ; then $\Omega(x, y)$, $\forall x, y \in K$; then $\Omega(x, y) \in K$

(ii) **Associative Property:** For any elements x, y and z of K , $\forall x, y, z \in K$; then

$$\Omega(x, \Omega(y, z)) = \Omega(\Omega(x, y), z) \tag{3.1}$$

(iii) **Identity Property :** K contains a unique element called identity element I such that for any element x of K , there exist an identity element $I \in K$ such that : $\forall x \in K$; then

$$\Omega(I, x) = \Omega(x, I) = x \tag{3.2}$$

(iv) **Inverse Property:** For any element x of K there is a unique element in K called inverse element x^{-1} such that $\forall x \in K$; $\exists$ inverse element $x^{-1} \in K$; then

$$\Omega(x^{-1}, x) = \Omega(x, x^{-1}) = I \tag{3.3}$$

The number of elements per group is either finite or infinite. The integers under the ordinary addition is true such that

$$y \sqcup x = x \sqcup y \tag{3.4}$$

for all pairs of integers x, y . Under the operation $\sqcup$, any two particular elements in a group satisfying (3.4) commute and the group is an Abelian if they are all pairs of elements in a group.

Definition 2: [A Group of Transformations]

Consider a transformations set:

$$x = (x_1, x_2, ..., x_m) \tag{3.5}$$

lie in a region $D \subset \mathbb{R}^m$. Consider the transformations set:

$$x^* = X(x, \varepsilon) \tag{3.6}$$

defined for each x in $D \subset \mathbb{R}$ depending on real parameter ε lying in $S \subset \mathbb{R}$. Suppose $\Omega(\varepsilon, \delta)$ defines a composition parameter law ε, δ then (3.6) forms a transformation group on D (Bluman and Anco, 2002).

$$x^* = X(x, \varepsilon) \tag{3.7}$$

$$x^{**} = X(x, \Omega(\varepsilon, \delta)) \tag{3.8}$$

Hence,it is a transformations Lie group.

Conditions for Transformations Lie Group

(i) for each parameter ε in S are one-to-one and onto D

(ii) S with composition law Ω forms a group

(iii) $x^* = x$ when $\varepsilon = I$,

$$X(x, I) = x \tag{3.9}$$

(iv) If $x^* = X(x, \varepsilon)$;

$$x^{**} = X(x, \Omega(\varepsilon, \delta)) \tag{3.10}$$

(v) ε is a continuous parameter and S is an interval in R

(vi) X is differentiable with respect to x infinitely in D and an analytic function of ε in S

(vii) $\Omega(\varepsilon, \delta)$ is analytic function of ε and δ, and $\varepsilon \in S, \delta \in S$

Illustration of Lie Group of Transformations (I)

Given that

$$x^* = X(x, \varepsilon) = x + \varepsilon \tag{3.11}$$

define a transformation.

$$x^* = x + \varepsilon \tag{3.12}$$

Here we see that $D = \mathbb{R}, S = \mathbb{R}$

$$x^* = X(x, \varepsilon) = X(x, 0) = x \tag{3.13}$$

$$x^{**} = X(x^*, \delta) = x^* + \delta = x + (\varepsilon + \delta) = x + \Omega(\varepsilon, \delta) \tag{3.14}$$

and clearly $x^* = X(x, \varepsilon)$ defines a simple group on K . Here $x^* = X(x, \varepsilon)$ is a Lie group of transformations.

Illustration of Lie Group of Transformations (II)

Consider

$$X(x, y, \varepsilon) = [x + \varepsilon, \frac{xy}{x + \varepsilon}] \tag{3.15}$$

$x^* = x + \varepsilon,$

$$y^* = \frac{xy}{x + \varepsilon} \tag{3.16}$$

$x^{**} = X(x^*, \delta) = x^* + \delta = x + (\varepsilon + \delta) = x + \Omega(\varepsilon, \delta)$

$$y^{**} = \left(\frac{x^* y^*}{x^* + \delta}\right) = xy(x + \Omega(\varepsilon, \delta)) \tag{3.17}$$

and

$$X(x, y, 0) = (x, y) \tag{3.18}$$

Hence the transformation $X(x, y, \varepsilon)$ forms a Lie group of transformations which has been widely used in this work when carrying out some operations onto some given equations.

3.3 Lie Algebras

Definition 3: [Vector Field]

A vector field V on a set M assigns a tangent vector $V|_x$ to each point $x \in M$ with $V|_x$ varying smoothly from point to point. In local coordinates $(x^1, x^2, ..., x^m)$ a vector field has the form:

$$V|_x = \xi^1(x)\frac{\partial}{\partial x^1} + \xi^2(x)\frac{\partial}{\partial x^2} + ... + \xi^m(x)\frac{\partial}{\partial x^m} \tag{3.19}$$

where each $\xi^i(x)$ is a smooth function of x (Olver, 1993).

Definition 4: [Commutator]

If G_1 and G_2 are vector fields then their commutator (also known as a Lie bracket) is defined as follows (Cantwell, 2002):

$$[G_1, G_2] = G_1G_2 - G_2G_1 \tag{3.20}$$

Illustration of Non-zero Lie Brackets

Consider the following two vector fields (Hydon, 2000):

$$G_1 = \frac{\partial}{\partial x}$$

$$G_2 = x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y}$$

The commutator for the two vector fields is:

$$[G_1, G_2] = \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y}\right) - \left(x\frac{\partial}{\partial x} + \frac{3}{4}y\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} = \frac{\partial}{\partial x} = G_1 \tag{3.21}$$

This was applied when calculating for non - zero Lie brackets which was very important in leading to the mathematical and general forms of the solution.

Definition 5:[Lie Algebra]

L, Lie algebra is a vector space over some field F, on which commutation is defined satisfying the following Sophus Lie conditions (Hydon, 2000):

(i) **Closure :**

$G_1, G_2 \in L \Longrightarrow [G_1 G_2] \in L$

(ii) **Skew-symmetry :**

$$[G_1, G_2] = -[G_2, G_1] \tag{3.22}$$

(iii) **Bi-linearity :**

$$[k_1 G_1 + k_2 G_2, G_3] = k_1[G_1, G_3] + k_2[G_2, G_3] \tag{3.23}$$

$$[G_1, k_1 G_2 + k_2 G_3] = k_1[G_1, G_2] + k_2[G_1, G_3] \tag{3.24}$$

where k_1 and k_2 are constants.

(iv) **Jacobi's Identity :**

$$[G_1, [G_2, G_3]] + [G_2, [G_3, G_1]] + [G_3, [G_1, G_2]] = 0 \tag{3.25}$$

for all G_1, G_2 and G_3 in L . If

$[G_1, G_2] = 0$, then it is said that G_1 and G_2 commute and if all the elements of L commute then L is called Abelian Lie algebra.

Definition 6:[Solvable Lie Algebra]

A solvable Lie algebra L has the derived series:

$L \supseteq L' = [L, L]$

$L \supseteq L'' = [L', L']$

$\vdots$

$L \supseteq L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$ such that $L^{(k)} = (0)$ for some $k > 0$ (Omolo, 1997).

3.4 Infinitesimal Transformations

Let us consider a transformation of one-parameter:

$$x^* = X(x, y, \lambda) \tag{3.26}$$

$$y^* = Y(x, y, \lambda) \tag{3.27}$$

where λ is a continuous parameter. By taking the Taylor series expansion of this transformation about the point $\lambda = \lambda_0$ generates :

$$x^* = x + \left(\frac{\partial X}{\partial \lambda}\right)_{\lambda=\lambda_0} (\lambda - \lambda_0) + ... \tag{3.28}$$

$$y^* = y + \left(\frac{\partial Y}{\partial \lambda}\right)_{\lambda=\lambda_0} (\lambda - \lambda_0) + \cdots \tag{3.29}$$

The partial derivatives evaluated at $\lambda = \lambda_0$ with respect to group parameter λ are known as infinitesimals (Cantwell, 2002) and are functions of x and y. Lets denote them by:

$$\left(\frac{\partial X}{\partial \lambda}\right)_{\lambda=\lambda_0} = \xi(x, y) \tag{3.30}$$

$$\left(\frac{\partial Y}{\partial \lambda}\right)_{\lambda=\lambda_0} = \eta(x, y) \tag{3.31}$$

Considering the values of λ sufficiently close to λ_0 by writing the coordinates of the transformation as follows:

$$x^* = x + \xi(x,y)(\lambda - \lambda_0) \tag{3.32}$$

$$y^* = y + \eta(x,y)(\lambda - \lambda_0) \tag{3.33}$$

where terms of second and higher degree in $(\lambda - \lambda_0)$ have been neglected. This transformation is known as an infinitesimal transformation (Dresner, 1999).

Infinitesimal Generators

The one-parameter Lie group of transformations of infinitesimal generator is an operator:

$$X = X(x) = \gamma(x).\nabla = \sum_{i=1}^{n} \gamma_i(x) \frac{\partial}{\partial x_i} \tag{3.34}$$

where the gradient operator ∇ is;

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n} \right) \tag{3.35}$$

for any function that is differentiable:

$$F(x) = F(x_1, x_2, ..., x_n) \tag{3.36}$$

$$XF(x) = \gamma(x).\nabla F(x) = \sum_{i=1}^{n} \gamma_i(x) \frac{\partial F(x)}{\partial x_i} \tag{3.37}$$

Thus a one - parameter transformations of Lie group is equivalent to its infinitesimal generator in the same way it is equivalent to its infinitesimal transformation.

Theorem of Transformations for One-parameter Lie Group

The transformations for one-parameter lie group is equal to:

$$x^* = e^{\varepsilon X} x = x + \varepsilon X x + \frac{\varepsilon^2}{2} X^2 x + ... \tag{3.38}$$

$$= [1 + \varepsilon X + \frac{\varepsilon^2}{2} X^2 + ...]x$$

$$= \sum_{i=0}^{\infty} \frac{\varepsilon^k}{k!} X^k x$$

where the generator $X = X(x)$ is the operator below defined by (3.34):

$$X^k = XX^{k-1}, k = 1, 2, ... \tag{3.39}$$

in particular $X^k F(x)$ is the function found by applying the operator X to the function $X^{k-1}F(x), k = 1, 2, ...$

3.5 Prolongations (Extended Transformations)

When applying a transformations point:

$$x^* = X(x, y, \omega) \tag{3.40}$$

$$y^* = Y(x, y, \omega) \tag{3.41}$$

to the differential equation:

$$H(x, y, y', y'', y''', ..., y^{(m)}) = 0 \tag{3.42}$$

$$y' = \frac{dy}{dx} \tag{3.43}$$

To transform the derivatives $y^{(m)}$ that is to extend (prolong) the point transformation to the derivatives. The task here is extending on the transformation (3.42) acting on (x, y) to the $(x, y, y_1, y_2, y_3, ..., y_m)$ space with the property of preserving the contact of differentials conditions:

$$dx, dy, dy_1, dy_2, ..., dy_m$$

$$\begin{aligned} dy &= y_1 dx \\ dy_1 &= y_2 dx \\ dy_2 &= y_3 dx \\ &\vdots \\ dy_m &= y_{m+1} dx \end{aligned} \tag{3.44}$$

3.6 Invariance under Transformations

Definition 7:[Invariant]

An invariant is that which remains unchanged when its constituents change. The concept of invariance has a physical basis in the conservation laws of mechanics. A function f under a Lie group is invariant iff;

$$f(x^*, y^*) = f(X(x, y, \lambda), Y(x, y, \lambda)) = f(x, y) \tag{3.45}$$

The function must read the same when expressed in the new variables (Cantwell, 2002). A simple example of invariance under a continuous transformation is the rotation of a circle about an axis that is normal to its centre.

3.7 Variation Symmetries

Definition 8:[Symmetry]

Symmetry is an operation which leaves invariant that upon which it operates. Symmetry of a transformation geometrical object apparently leaves the object unchanged. Consider the transformation of infinitesimal form:

$$x_i^* = x_i + \varepsilon\omega_i, i = 1, 2, ..., m \tag{3.46}$$

where ε is a parameter of smallness. Here equation (3.46) can be written as

$$x_i^* = (1 + \varepsilon G)x_i \tag{3.47}$$

where

$$G = \omega_i \frac{\partial}{\partial x_i} \tag{3.48}$$

is a differential operator called the generator of the transformation (3.46). Consider:

$$G = \omega\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} \tag{3.49}$$

Under the infinitesimal transformation generated by G, a function $f(x, y)$ becomes:

$$f^*(x^*, y^*) = (1 + \varepsilon G)f(x, y)$$

$$= f + \varepsilon \left(\omega \frac{\partial f}{\partial x} + \phi \frac{\partial f}{\partial y} \right) \tag{3.50}$$

If the form of f is unchanged then

$$f^*(x^*, y^*) = f(x, y) \tag{3.51}$$

or

$$\omega \frac{\partial f}{\partial x} + \phi \frac{\partial f}{\partial y} = 0 \tag{3.52}$$

then G is called a symmetry of f. In mathematics, all symmetries represent invariance under transformations which may be translations, reflections or rotations.

3.8 Lie Theory of Differential Equations

Definition 9:[Lie Point Symmetries of ODEs]

Point symmetry is a symmetry in which the infinitesimals depend only on coordinates (Yulia, 2008). Lie point symmetry is described as a point symmetry that depends continuously on at least one-parameter, thus the parameters can vary continuously over a set of scalar non-zero measure. Lie point symmetries of ODEs are of the form:

$$G = \omega \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y} \tag{3.53}$$

where ω and ϕ are coefficients functions of only x and y. To apply a point transformation to an m^{th} order ODE;

$$f(x, y, y', y'', ..., y^{(m)}) = 0 \tag{3.54}$$

where:

$$y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}, ..., y^{(m)} = \frac{d^m y}{dx^m} \tag{3.55}$$

There is a need to know how derivatives undergo the infinitesimal transformation:

$$x^* = x + \varepsilon\omega(x, y) \tag{3.56}$$

$$y^* = y + \varepsilon\phi(x, y) \tag{3.57}$$

which has a generator given by

$$G = \omega(x,y)\frac{\partial}{\partial x} + \phi(x,y)\frac{\partial}{\partial y} \tag{3.58}$$

In terms of the quantities x^* and y^* it gives the derivative;

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(y+\varepsilon\phi)}{d(x+\varepsilon\omega)} \\ &= \frac{\frac{dy}{dx}+\varepsilon\frac{d\phi}{dx}}{1+\varepsilon\frac{d\omega}{dx}} \\ &= (y'+\varepsilon\phi')(1-\varepsilon\omega'+\varepsilon^2\omega'^2-...) \\ &= y'+\varepsilon(\phi'-y'\omega') \end{aligned} \tag{3.59}$$

which was terminated at $O(\varepsilon^2)$. The primes here are for total differentiation with respect to x. Now the second derivative gives:

$$\begin{aligned} \frac{d^2y^*}{dx^{*2}} &= \frac{d}{dx^*}\left(\frac{dy^*}{dx^*}\right) \\ &= \frac{d[y'+\varepsilon(\phi'-y'\omega')]}{d(x+\varepsilon\omega)} \\ &= \frac{\frac{dy'}{dx}+\varepsilon\frac{d}{dx}(\phi'-y'\omega')}{1+\varepsilon\omega'} \\ &= y''+\varepsilon(\phi''-2y''\omega'-y'\omega'') \end{aligned} \tag{3.60}$$

Further, the third derivative is as follows:

$$\frac{d^3y^*}{dx^{*3}} = y'''+\varepsilon(\phi'''-3y'''\omega'-3y''\omega''-y'\omega''') \tag{3.61}$$

A fourth derivative yields:

$$\frac{d^4y^*}{dx^{*4}} = y^{(iv)}+\varepsilon(\phi^{(iv)}-4y^{(iv)}\omega'-6y'''\omega''-4y''\omega'''-y'\omega^{(iv)}) \tag{3.62}$$

In general, it generates the formula (Leach, *et. al*,2007):

$$\frac{d^my^*}{dx^{*m}} = y^{(m)}+\varepsilon\left(\phi^{(m)}-\sum_{i=1}^{m}C_i^m y^{(i+1)}\omega^{(m-i)}\right) \tag{3.63}$$

where the superscript (i) denotes $\frac{d^i}{dx^i}$ and C_i^m is the number of combinations of m objects taken i at a time. To deal with the infinitesimal transformations of equations and functions involving derivatives, the extensions of the generator G are needed. Its indicated that a generator G has been extended by writing

$$G^{[1]} = G + (\phi' - y'\omega')\frac{\partial}{\partial y'} \tag{3.64}$$

$$G^{[2]} = G^{[1]} + (\phi'' - 2y''\omega' - y'\omega'')\frac{\partial}{\partial y''} \tag{3.65}$$

for the first and second extensions respectively. When generating an extension of G it has to extend G such that all derivatives appearing in the equation or function are included in the extension. For an m^{th} order differential equation, the m^{th} extension is of the form (Bluman, *et. al,* 2009):

$$G^{[m]} = G + \sum_{i=1}^{m}\left\{\phi^{(i)} - \sum_{j=1}^{i}\binom{i}{j} y^{(i+1-j)}\omega^{(i)}\right\}\frac{\partial}{\partial y^{(i)}} \tag{3.66}$$

The generator

$$G = \omega\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y}$$

is a symmetry of the differential equation

$$E(x, y, y', y'', ..., y^{(m)}) = 0 \tag{3.67}$$

if and only if

$$G^{[m]}E_{E=0} = 0 \tag{3.68}$$

which means that the action of the m^{th} extension of G on E is zero when the original equation is satisfied.

3.9 Reduction of Order

If a differential equation:

$$E(x, y, y', ..., y^{(m)}) = 0 \tag{3.69}$$

has a symmetry:

$$G = \omega(x,y)\frac{\partial}{\partial x} + \phi(x,y)\frac{\partial}{\partial y} \tag{3.70}$$

to obtain an equation of order $(m-1)$ in a systematic manner. This is achieved by using the $zero^{th}$ and first order differential invariants which are the two characteristics associated with $G^{[1]}$. The characteristics are obtained by solving the following system of ODEs (Baumann, 2000):

$$\frac{dx}{\omega} = \frac{dy}{\phi} = \frac{dy'}{(\phi' - y'\omega')} \tag{3.71}$$

3.10 Integrating Factors

Integrating factors are all solutions of both the adjoint symmetry of the linearised system of ordinary differential equations and a system that represents an extra-adjoint-invariance condition. The following theorem establishes the relationship between integrating factors and infinitesimal symmetries of differential equations of the first order.

Theorem of Integrating Factor

Consider a first order ODE:

$$M(x,y)dx + N(x,y)dy = 0 \tag{3.72}$$

which admits a one-parameter Lie group G with an infinitesimal generator:

$$X = \sigma(x,y)\frac{\partial}{\partial x} + \psi(x,y)\frac{\partial}{\partial y} \tag{3.73}$$

if and only if the function:

$$\rho = \frac{1}{(\sigma M + \psi N)} \tag{3.74}$$

is the integrating factor for equation (3.72) provided that $\sigma M + \psi N \neq 0$.

CHAPTER 4

RESULTS AND DISCUSSION

4.1 Introduction

This study determined a solution to a third order first degree nonlinear, non-homogeneous ODE of fourth degree in second derivative which is a form of a wave equation:

$$F(x, y, y', y'', y''') = 0 \quad \text{or} \quad y''' = f(x, y, y', y'') \tag{4.1}$$

The objectives were to develop and determine both mathematical and general solutions to the special case (4.1) of the form:

$$y''' - y'\left(\frac{y''}{y}\right)^4 = 0 \tag{4.2}$$

using the method of Lie symmetry. By expressing (4.2) in other ways gives:

$$y''' - y'(\frac{y''^4}{y^4}) = 0$$

when the power is brought into the bracket.

$$\Rightarrow y''' - y'(y)^{-4}(y'')^4 = 0 \tag{4.3}$$

after applying the law of indices and removal of the fraction hence known as the transformation equation.

4.2 Mathematical Solution of Nonlinear Wave Equation of Third Order

By applying the m^{th} extension of G given as:

$$G^{[m]} = G + \sum_{i=1}^{m} \left\{ \phi^{(i)} - \sum_{j=1}^{i} \binom{i}{j} y^{(i+1-j)} \omega^{(i)} \right\} \frac{\partial}{\partial y^{(i)}} \tag{4.4}$$

where m is the order, i is the upper limit and j is the lower limit, from (3.66). Then the third extension of $G^{[3]}$ is:

$$\begin{aligned} G^{[3]} &= G^{[2]} + (\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega''')\frac{\partial}{\partial y'''} \\ &= G^{[1]} + (\phi'' - 2y''\omega' - y'\omega'')\frac{\partial}{\partial y''} + (\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega''')\frac{\partial}{\partial y'''} \end{aligned}$$

$$\begin{aligned}\therefore G^{[3]} &= \omega\frac{\partial}{\partial x}+\phi\frac{\partial}{\partial y}+(\phi'-\omega' y')\frac{\partial}{\partial y'}+(\phi''-2y''\omega'-y'\omega'')\frac{\partial}{\partial y''}\\ &+ (\phi'''-3y'''\omega'-3y''\omega''-y'\omega''')\frac{\partial}{\partial y'''} \qquad (4.5)\end{aligned}$$

Now, manipulating $G^{[3]}$ on (4.3) yields:

$$G^{[3]}\left(y'''-y'(y'')^4(y)^{-4}\right)=0 \qquad (4.6)$$

$$\begin{aligned}&\Rightarrow [\omega\frac{\partial}{\partial x}+\phi\frac{\partial}{\partial y}+(\phi-\omega' y')\frac{\partial}{\partial y'}+(\phi''-2y''\omega'-y'\omega'')\frac{\partial}{\partial y''}\\ &+ (\phi'''-3y'''\omega'-3y''\omega''-y'\omega''')\frac{\partial}{\partial y'''}](y'''-y'(y'')^4(y)^{-4})\\ &= 0 \qquad (4.7)\end{aligned}$$

Hence, expansion of (4.7) gives:

$$\begin{aligned}&\Rightarrow \omega\frac{\partial}{\partial x}\left(y'''-y'(y'')^4(y)^{-4}\right)+\phi\frac{\partial}{\partial y}\left(y'''-y'(y'')^4(y)^{-4}\right)\\ &+ (\phi'-\omega' y')\frac{\partial}{\partial y'}\left[y'''-y'(y'')^4(y)^{-4}\right]\\ &+ (\phi''-2y''\omega'-y'\omega'')\frac{\partial}{\partial y''}\left[y'''-y'(y'')^4(y)^{-4}\right]\\ &+ (\phi'''-3y'''\omega'-3y''\omega''-y'\omega''')\frac{\partial}{\partial y'''}[y'''-y'(y'')^4(y)^{-4}]=0 \qquad (4.8)\end{aligned}$$

From (4.8) it implies:

$$\begin{aligned}&\omega\frac{\partial}{\partial x}[y'''-y'(y'')^4(y)^{-4}](4.8a)\\ &= \omega[y^{iv}-y''(y'')^4(y)^{-4}-4(y'')^3y'''y'(y)^{-4}+4(y)^{-5}y'y'(y'')^4]\\ &= \omega[y^{iv}-y''(y'')^4(y)^{-4}-4(y'')^3y'''y'(y)^{-4}+4(y)^{-5}y'y'(y'')^4]\\ &= \omega[y^{(iv)}-(y'')^5(y)^{-4}-4y'(y'')^3(y)^{-4}y'''+4(y')^2(y'')^4(y)^{-5}]\\ &\phi\frac{\partial}{\partial y}[y'''-y'(y'')^4(y)^{-4}](4.8b)\\ &= \phi[0-(0+0-4(y)^{-5}y'(y'')^4)]\\ &= \phi[0+4(y)^{-5}y'(y'')^4]\\ &= \phi[4y'(y'')^4(y)^{-5}]\\ &(\phi'-\omega' y')\frac{\partial}{\partial y'}[y'''-y'(y'')^4(y)^{-4}](4.8c)\end{aligned}$$

$$
\begin{aligned}
&= (\phi' - \omega' y')[0 - y''(y'')^4(y)^{-4}] \\
&= (\phi' - \omega' y')[-(y'')^5(y)^{-4}] \\
&\quad (\phi'' - 2y''\omega' - y'\omega'')\frac{\partial}{\partial y''}[y''' - y'(y'')^4(y)^{-4}](4.8d) \\
&= (\phi'' - 2y''\omega' - y'\omega'')[0 - 4(y'')^3 y''' y'(y)^{-4}] \\
&= (\phi'' - 2y''\omega' - y'\omega'')[-4y'(y'')^3(y)^{-4}y'''] \\
&\quad (\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega''')\frac{\partial}{\partial y'''}[y''' - y'(y'')^4(y)^{-4}](4.8e) \\
&= (\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega''')[1 - 0] \\
&= (\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega''')
\end{aligned}
$$

By combining (4.8a) to (4.8e) gives:

$$
\begin{aligned}
&\omega[y^{(iv)} - (y'')^5(y)^{-4} - 4y'(y'')^3(y)^{-4}y''' + 4(y')^2(y'')^4(y)^{-5}] \\
+ \; &\phi[4y'(y'')^4(y)^{-5}] + (\phi' - \omega' y')[-(y'')^5(y)^{-4}] \\
+ \; &(\phi'' - 2y''\omega' - y'\omega'')[-4y'(y'')^3(y)^{-4}y'''] \\
+ \; &(\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega''') = 0
\end{aligned}
\tag{4.9}
$$

Thus, from (4.3) gives:

$y''' - y'(y'')^4(y)^{-4} = 0$

$$\Rightarrow y''' = y'(y'')^4(y)^{-4} \tag{4.10}$$

and

$y^{(iv)} = (y''')'$

hence

$y^{(iv)} = (y'(y'')^4(y)^{-4})' = y''(y'')^4(y)^{-4} + 4y'(y'')^3y'''(y)^{-4} - 4(y)^{-5}y'y'(y'')^4$

$$y^{(iv)} = (y'')^5(y)^{-4} + 4y'(y'')^3y'''(y)^{-4} - 4(y')^2(y'')^4(y)^{-5} \tag{4.11}$$

By putting (4.11) into (4.9) gives:

$$\omega[(y'')^5(y)^{-4} + 4y'(y'')^3y'''(y)^{-4} - 4(y')^2(y'')^4(y)^{-5} - (y'')^5(y)^{-4}$$

$$
\begin{aligned}
- &\quad 4y'(y'')^3(y)^{-4}y''' + 4(y')^2(y'')^4(y)^{-5}] \\
+ &\quad \phi[4y'(y'')^4(y)^{-5}] + (\phi' - \omega'y')[-(y'')^5(y)^{-4}] \\
+ &\quad (\phi'' - 2y''\omega' - y'\omega'')[-4y'(y'')^3(y)^{-4}y'''] \\
+ &\quad (\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega''') = 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow &\quad \omega[(y'')^5(y)^{-4} + 4y'(y'')^3y'''(y)^{-4} - 4(y')^2(y'')^4(y)^{-5} - (y'')^5(y)^{-4} \\
- &\quad 4y'(y'')^3(y)^{-4}y''' + 4(y')^2(y'')^4(y)^{-5}] \\
+ &\quad [4y'(y'')^4(y)^{-5}]\phi - [(y'')^5(y)^{-4}](\phi' - \omega'y') \\
- &\quad [4y'(y'')^3(y)^{-4}y'''](\phi'' - 2y''\omega' - y'\omega'') \\
+ &\quad (\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega''') = 0 \qquad (4.12)
\end{aligned}
$$

Further simplification gives:

$$
\begin{aligned}
&\quad \omega(y'')^5(y)^{-4} + 4\omega y'(y'')^3(y)^{-4}y''' - 4\omega(y')^2(y'')^4(y)^{-5} \\
- &\quad \omega(y'')^5(y)^{-4} - 4\omega y'(y'')^3(y)^{-4}y''' + 4\omega(y')^2(y'')^4(y)^{-5} \\
+ &\quad 4\phi y'(y'')^4(y)^{-4} - \phi'(y'')^5(y)^{-4} + \omega'y'(y'')^5(y)^{-4} \\
- &\quad 4\phi''y'(y'')^3(y)^{-4}y''' + 8\omega'y'(y'')^4(y)^{-4}y''' \\
+ &\quad 4\omega''(y')^2(y'')^3(y)^{-4}y''' \\
+ &\quad \phi''' - 3\omega'y''' - 3\omega''y'' - y'\omega''' = 0 \qquad (4.13)
\end{aligned}
$$

Again,

$$
\begin{aligned}
&\omega(y)^{-4}(y'')^5 + 4\omega y'(y)^{-4}(y'')^3y''' - 4\omega(y)^{-5}(y')^2(y'')^4 - \omega(y)^{-4}(y'')^5 \\
&-4\omega y'(y)^{-4}(y'')^3y''' + 4\omega(y)^{-5}(y')^2(y'')^4 + 4\phi y'(y)^{-4}(y'')^4 \\
&-\phi'(y)^{-4}(y'')^5 + \omega'y'(y)^{-4}(y'')^5 - 4\phi''y'(y)^{-4}(y'')^3y''' \\
&+8\omega'y'(y)^{-4}(y'')^4y''' + 4\omega''(y)^{-4}(y')^2(y'')^3y''' \\
&+\phi''' - 3\omega'y''' - 3\omega''y'' - y'\omega''' = 0
\end{aligned}
$$

When simplified, it gives:

$$
4\phi y'(y)^{-4}(y'')^4 - \phi'(y)^{-4}(y'')^5 - 4\phi''y'(y)^{-4}(y'')^3y''' + \phi'''
$$

$$
\begin{aligned}
&+\quad 8\omega'y'(y)^{-4}(y'')^4y''' + \omega'y'(y)^{-4}(y'')^5 - 3\omega'y''' - 3\omega''y'' \\
&+\quad 4\omega''(y)^{-4}(y')^2(y'')^3y''' - y'\omega''' = 0
\end{aligned} \tag{4.14}
$$

By expressing first, second and third derivatives of ω and ϕ in terms of partial derivatives given that:

$\omega = \omega(x, y)$

then

$d(\omega) = \left(\frac{\partial\omega}{\partial x}\right) dx + \left(\frac{\partial\omega}{\partial y}\right) dy$

$$
\therefore \omega' = \frac{\partial\omega}{\partial x} + y'\frac{\partial\omega}{\partial y} \tag{4.15}
$$

$\omega'' = \frac{d}{dx}(\omega') + \frac{d}{dy}(\omega')y'$

$\omega'' = \frac{d}{dx}\left(\frac{\partial\omega}{\partial x} + y'\frac{\partial\omega}{\partial y}\right) + \frac{d}{dy}\left(\frac{\partial\omega}{\partial x} + y'\frac{\partial\omega}{\partial y}\right) y'$

$= \frac{\partial^2\omega}{\partial x^2} + y'\frac{\partial^2\omega}{\partial x\partial y} + y''\frac{\partial\omega}{\partial y} + y'\frac{\partial^2\omega}{\partial x\partial y} + y'^2\frac{\partial^2\omega}{\partial y^2} + 0$

Simplification yields:

$$
\therefore \omega'' = 2y'\frac{\partial^2\omega}{\partial x\partial y} + y'^2\frac{\partial^2\omega}{\partial y^2} + y''\frac{\partial\omega}{\partial y} + \frac{\partial^2\omega}{\partial x^2} \tag{4.16}
$$

$\omega''' = \frac{d}{dx}(\omega'') + y'\frac{d}{dy}(\omega'')$

$$
\begin{aligned}
\omega''' \quad &= \quad \frac{d}{dx}\left(\frac{\partial^2\omega}{\partial x^2} + 2y'\frac{\partial^2\omega}{\partial x\partial y} + y''\frac{\partial\omega}{\partial y} + y'^2\frac{\partial^2\omega}{\partial y^2}\right) \\
&+ \quad y'\frac{d}{dy}\left(\frac{\partial^2\omega}{\partial x^2} + 2y'\frac{\partial^2\omega}{\partial x\partial y} + y''\frac{\partial\omega}{\partial y} + y'^2\frac{\partial^2\omega}{\partial y^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \quad \frac{\partial^3\omega}{\partial x^3} + 2y'\frac{\partial^3\omega}{\partial x^2\partial y} + 2y''\frac{\partial^2\omega}{\partial x\partial y} + y'''\frac{\partial\omega}{\partial y} \\
&+ \quad 2y'y''\frac{\partial^2\omega}{\partial y^2} + y'\frac{\partial^3\omega}{\partial x^2\partial y} + y'^2\frac{\partial^3\omega}{\partial x\partial y^2} + 2y'^2\frac{\partial^3\omega}{\partial x\partial y^2} + 0 \\
&+ \quad y'y''\frac{\partial^2\omega}{\partial y^2} + 0 + y^3\frac{\partial^3\omega}{\partial y^3} + 0
\end{aligned}
$$

$$\therefore \omega''' = \frac{\partial^3\omega}{\partial x^3} + 3y'\frac{\partial^3\omega}{\partial x^2\partial y} + 3y''\frac{\partial^2\omega}{\partial x\partial y} + y'''\frac{\partial\omega}{\partial y} + 3y''y'\frac{\partial^2\omega}{\partial y^2} + y'^3\frac{\partial^3\omega}{\partial y^3} + 3y'^2\frac{\partial^3\omega}{\partial x\partial y^2} \tag{4.17}$$

and also

$\phi = \phi(x, y)$

then

$d(\phi) = \left(\frac{\partial\phi}{\partial x}\right) dx + \left(\frac{\partial\phi}{\partial y}\right) dy$

$$\therefore \phi' = \frac{\partial\phi}{\partial x} + y'\frac{\partial\phi}{\partial y} \tag{4.18}$$

$\phi'' = \frac{d}{dx}(\phi') + \frac{d}{dy}(\phi')y'$

$\phi'' = \frac{d}{dx}\left(\frac{\partial\phi}{\partial x} + y'\frac{\partial\phi}{\partial y}\right) + \frac{d}{dy}\left(\frac{\partial\phi}{\partial x} + y'\frac{\partial\phi}{\partial y}\right)y'$

$= \frac{\partial^2\phi}{\partial x^2} + y'\frac{\partial^2\phi}{\partial x\partial y} + y''\frac{\partial\phi}{\partial y} + y'\frac{\partial^2\phi}{\partial x\partial y} + y'^2\frac{\partial^2\phi}{\partial y^2} + 0$

Simplification yields:

$$\therefore \phi'' = 2y'\frac{\partial^2\phi}{\partial x\partial y} + y'^2\frac{\partial^2\phi}{\partial y^2} + y''\frac{\partial\phi}{\partial y} + \frac{\partial^2\phi}{\partial x^2} \tag{4.19}$$

$\phi''' = \frac{d}{dx}(\phi'') + y'\frac{d}{dy}(\phi'')$

$= \frac{d}{dx}\left(\frac{\partial^2\phi}{\partial x^2} + 2y'\frac{\partial^2\phi}{\partial x\partial y} + y''\frac{\partial\phi}{\partial y} + y'^2\frac{\partial^2\phi}{\partial y^2}\right) + y'\frac{d}{dy}\left(\frac{\partial^2\phi}{\partial x^2} + 2y'\frac{\partial^2\phi}{\partial x\partial y} + y''\frac{\partial\phi}{\partial y} + y'^2\frac{\partial^2\phi}{\partial y^2}\right)$

$$= \frac{\partial^3\phi}{\partial x^3} + 2y'\frac{\partial^3\phi}{\partial x^2\partial y} + 2y''\frac{\partial^2\phi}{\partial x\partial y} + y'''\frac{\partial\phi}{\partial y} + y'^2\frac{\partial^3\phi}{\partial x\partial y^2} + 2y'y''\frac{\partial^2\phi}{\partial y^2} + y''\frac{\partial^2\phi}{\partial x\partial y} + y'\frac{\partial^3\phi}{\partial x^2\partial y} + 0 + 2y'^2\frac{\partial^3\phi}{\partial x\partial y^2} + 0 + y'y''\frac{\partial^2\phi}{\partial y^2} + 0 + y'^3\frac{\partial^3\phi}{\partial y^3} + 0$$

$$\therefore \phi''' = \frac{\partial^3\phi}{\partial x^3} + 3y'\frac{\partial^3\phi}{\partial x^2\partial y} + 3y''\frac{\partial^2\phi}{\partial x\partial y} + y'''\frac{\partial\phi}{\partial y} + 3y'^2\frac{\partial^3\phi}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\phi}{\partial y^2} + y'^3\frac{\partial^3\phi}{\partial y^3} \tag{4.20}$$

By substituting (4.15), (4.16), (4.18), (4.19), (4.20) into (4.14) gives:

$$\begin{aligned} & 4\phi y'(y)^{-4}(y'')^4 - \phi'(y)^{-4}(y'')^5 - 4\phi'' y'(y)^{-4}(y'')^3 y''' + \phi''' \\ + \quad & 8\omega' y'(y)^{-4}(y'')^4 y''' + \omega' y'(y)^{-4}(y'')^5 - 3\omega' y''' - 3\omega'' y'' \\ + \quad & 4\omega''(y)^{-4}(y')^2(y'')^3 y''' - y'\omega''' = 0 \end{aligned} \tag{4.21}$$

Equally :

$$\begin{aligned} & 4\phi y'(y)^{-4}(y'')^4 - \left(\frac{\partial\phi}{\partial x} + y'\frac{\partial\phi}{\partial y}\right)(y)^{-4}(y'')^5 \\ - \quad & 4\left(\frac{\partial^2\phi}{\partial x^2} + 2y'\frac{\partial^2\phi}{\partial x\partial y} + y'^2\frac{\partial^2\phi}{\partial y^2} + y''\frac{\partial\phi}{\partial y}\right) y'(y)^{-4}(y'')^3 y''' \\ + \quad & \left(\frac{\partial^3\phi}{\partial x^3} + 3y'\frac{\partial^3\phi}{\partial x^2\partial y} + 3y''\frac{\partial^2\phi}{\partial x\partial y} + y'''\frac{\partial\phi}{\partial y} + 3y'^2\frac{\partial^3\phi}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\phi}{\partial y^2} + y'^3\frac{\partial^3\phi}{\partial y^3}\right) \\ + \quad & 8\left(\frac{\partial\omega}{\partial x} + y'\frac{\partial\omega}{\partial y}\right) y'(y)^{-4}(y'')^4 y''' + \left(\frac{\partial\omega}{\partial x} + y'\frac{\partial\omega}{\partial y}\right) y'(y)^{-4}(y'')^5 \\ - \quad & 3\left(\frac{\partial\omega}{\partial x} + y'\frac{\partial\omega}{\partial y}\right) y''' - 3\left(\frac{\partial^2\omega}{\partial x^2} + 2y'\frac{\partial^2\omega}{\partial x\partial y} + y'^2\frac{\partial^2\omega}{\partial y^2} + y''\frac{\partial\omega}{\partial y}\right) y'' \\ + \quad & 4\left(\frac{\partial^2\omega}{\partial x^2} + 2y'\frac{\partial^2\omega}{\partial x\partial y} + y'^2\frac{\partial^2\omega}{\partial y^2} + y''\frac{\partial\omega}{\partial y}\right)(y)^{-4}(y')^2(y'')^3 y''' \\ - \quad & y'\left(\frac{\partial^3\omega}{\partial x^3} + 3y'\frac{\partial^3\omega}{\partial x^2\partial y} + 3y''\frac{\partial^2\omega}{\partial x\partial y} + y'''\frac{\partial\omega}{\partial y} + 3y'^2\frac{\partial^3\omega}{\partial x\partial y^2} + 3y'y''\frac{\partial^2\omega}{\partial y^2} + y'^3\frac{\partial^3\omega}{\partial y^3}\right) \\ = \quad & 0 \end{aligned} \tag{4.22}$$

When expanded, it yields:

$$\begin{aligned} \therefore \quad & 4\phi y'(y)^{-4}(y'')^4 - (y)^{-4}(y'')^5\frac{\partial\phi}{\partial x} - (y)^{-4}(y'')^5 y'\frac{\partial\phi}{\partial y} - 4y'(y)^{-4}(y'')^3 y'''\frac{\partial^2\phi}{\partial x^2} \\ - \quad & 8(y')^2(y)^{-4}(y'')^3 y'''\frac{\partial^2\phi}{\partial x\partial y} - 4(y')^3(y)^{-4}(y'')^3 y'''\frac{\partial^2\phi}{\partial y^2} - 4y'(y)^{-4}(y'')^4 y'''\frac{\partial\phi}{\partial y} \\ + \quad & \frac{\partial^3\phi}{\partial x^3} + 3y'\frac{\partial^3\phi}{\partial x^2\partial y} + 3y''\frac{\partial^2\phi}{\partial x\partial y} + y'''\frac{\partial\phi}{\partial y} + 3(y')^2\frac{\partial^3\phi}{\partial x\partial y^2} + 3y''y'\frac{\partial^2\phi}{\partial y^2} + (y')^3\frac{\partial^3\phi}{\partial y^3} \\ + \quad & y'''\frac{\partial\phi}{\partial y} + 8y'(y)^{-4}(y'')^4 y'''\frac{\partial\omega}{\partial x} + 8(y')^2(y)^{-4}(y'')^4 y'''\frac{\partial\omega}{\partial y} \\ + \quad & y'(y)^{-4}(y'')^5\frac{\partial\omega}{\partial x} + (y')^2(y)^{-4}(y'')^5\frac{\partial\omega}{\partial y} - 3y'''\frac{\partial\omega}{\partial x} - 3y'''y'\frac{\partial\omega}{\partial y} \\ - \quad & 3y''\frac{\partial^2\omega}{\partial x^2} - 6y''y'\frac{\partial^2\omega}{\partial x\partial y} - 3y''(y')^2\frac{\partial^2\omega}{\partial y^2} - 3(y'')^2\frac{\partial\omega}{\partial y} \\ + \quad & 4(y)^{-4}(y')^2(y'')^3 y'''\frac{\partial^2\omega}{\partial x^2} + 8(y')^3(y)^{-4}(y'')^3 y'''\frac{\partial^2\omega}{\partial x\partial y} \end{aligned}$$

$$
\begin{aligned}
&+ \quad 4(y')^4(y)^{-4}(y'')^3y'''\frac{\partial^2\omega}{\partial y^2} + 4(y'')^4(y)^{-4}(y')^2y'''\frac{\partial\omega}{\partial y} \\
&- \quad y'\frac{\partial^3\omega}{\partial x^3} - 3(y')^2\frac{\partial^3\omega}{\partial x^2\partial y} - 3y'y''\frac{\partial^2\omega}{\partial x\partial y} - 3(y')^3\frac{\partial^3\omega}{\partial x\partial y^2} \\
&- \quad 3y''(y')^2\frac{\partial^2\omega}{\partial y^2} - (y')^4\frac{\partial^3\omega}{\partial y^3} - y'y''\frac{\partial\omega}{\partial y} = 0
\end{aligned} \tag{4.23}
$$

where (4.23) forms an identity in x, y, y', y'', y''' . Given that ω and ϕ are functions in x and y alone, by equating the combinations of coefficients of the powers of y', y'', y''' to zero, it yields:

$$(y')^4(y'')^3y''' : 4(y)^{-4}\frac{\partial^2\omega}{\partial y^2} = 0 \tag{4.24}$$

$$(y')^3(y'')^3y''' : 8(y)^{-4}\frac{\partial^2\omega}{\partial x\partial y} - 4(y)^{-4}\frac{\partial^2\phi}{\partial y^2} = 0 \tag{4.25}$$

$$(y')^2(y'')^3y''' : 4(y)^{-4}\frac{\partial^2\omega}{\partial x^2} - 8(y)^{-4}\frac{\partial^2\phi}{\partial x\partial y} = 0 \tag{4.26}$$

$$(y')^1(y'')^3y''' : -4(y)^{-4}\frac{\partial^2\phi}{\partial x^2} = 0 \tag{4.27}$$

By integrating (4.24):

$4(y)^{-4}\frac{\partial^2\omega}{\partial y^2} = 0$

$\Rightarrow \frac{\partial^2\omega}{\partial y^2} = 0$

$\frac{\partial\omega}{\partial y} = A_1$

$$\therefore \omega = A_1y + A_2 \tag{4.28}$$

where A_1 and A_2 are arbitrary functions of x . By substituting (4.28) into (4.25) and then solving gives;

$8(y)^{-4}\frac{\partial^2\omega}{\partial x\partial y} - 4(y)^{-4}\frac{\partial^2\phi}{\partial y^2} = 0$

$\Rightarrow 2\frac{\partial^2\omega}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y^2} = 0$

$2\frac{\partial A_1}{\partial x} - \frac{\partial^2\phi}{\partial y^2} = 0$

$\Rightarrow \frac{\partial^2\phi}{\partial y^2} = 2A_1'$

$\Rightarrow \frac{\partial\phi}{\partial y} = 2A_1'y + A_3$

$$\therefore \phi = A_1'y^2 + A_3y + A_4 \tag{4.29}$$

where A_3 and A_4 are arbitrary functions of x. By substituting (4.28), (4.29) into (4.26) yields:

$4(y)^{-4}\frac{\partial^2\omega}{\partial x^2} - 8(y)^{-4}\frac{\partial^2\phi}{\partial x\partial y} = 0$

$\Rightarrow \frac{\partial^2\omega}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\partial y} = 0$

$\Rightarrow -2\frac{\partial}{\partial x}(2A_1'y + A_3) + (A_1''y + A_2'') = 0$

$\Rightarrow -2(2A_1''y + A_3') + A_1''y + A_2'' = 0$

$\Rightarrow -4A_1''y - 2A_3' + A_1''y + A_2'' = 0$

$$\therefore 3A_1''y + 2A_3' - A_2'' = 0 \tag{4.30}$$

By equating the coefficients of powers of y^0 and y^1 to zero in (4.30):

$$y^1 : 3A_1'' = 0 \tag{4.31}$$

$$y^0 : 2A_3' - A_2'' = 0 \tag{4.32}$$

By substituting (4.28), (4.29) into (4.27):

$-4(y)^{-4}\frac{\partial^2\phi}{\partial y^2} = 0$

$-4(y)^{-4}\frac{\partial^2}{\partial y^2}(A_1'y^2 + A_3y + A_4) = 0$

$-4(y)^{-4}(A_1''y^2 + A_3''y + A_4'') = 0$

$$\therefore A_1''(y)^{-2} + A_3''(y)^{-3} + A_4''(y)^{-4} = 0 \tag{4.33}$$

By equating the coefficients of powers of y^{-4}, y^{-3} and y^{-2} to zero:

$$y^{-4} : A_4'' = 0 \tag{4.34}$$

$$y^{-3} : A_3'' = 0 \tag{4.35}$$

$$y^{-2} : A_1'' = 0 \tag{4.36}$$

By solving (4.31):

$\Rightarrow A_1'' = 0$

Then

$A_1' = B_1$

Hence

$$\therefore A_1 = B_1 x + B_2 \tag{4.37}$$

Now taking (4.35) and solving:

$\Rightarrow A_3'' = 0$

Then

$A_3' = B_3$

$$\therefore A_3 = B_3 x + B_4 \tag{4.38}$$

Taking (4.32) and solving:

$2A_3' - A_2'' = 0$

$\Rightarrow A_2'' = 2A_3'$

Thus

$A_2'' = 2B_3$ (Since $A_3' = B_3$)

$A_2' = 2B_3 x + B_5$

$$\therefore A_2 = B_3 x^2 + B_5 x + B_6 \tag{4.39}$$

From (4.34):

$\Rightarrow A_4'' = 0$

Thus

$A_4' = B_7$

$$\therefore A_4 = B_7 x + B_8 \tag{4.40}$$

where $B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8$ are arbitrary constants. From

$\omega = A_1 y + A_2$

and substituting A_1 and A_2 :

$\Rightarrow \omega = (B_1 x + B_2)y + (B_3 x^2 + B_5 x + B_6)$

$$\therefore \omega = B_1 xy + B_2 y + B_3 x^2 + B_5 x + B_6 \tag{4.41}$$

Again, substituting (4.34), (4.37), (4.40) into (4.29): From

$\phi = A_1' y^2 + A_3 y + A_4$

then by substituting A_1, A_3 and A_4 gives:

$$\therefore \phi = (B_1 x + B_2)' y^2 + (B_3 x + B_4)y + B_7 x + B_8 \tag{4.42}$$

Now the infinitesimal generator G is of the form:

$G = \omega \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y}$

By substituting ω and ϕ , this form is then given as:

$G = (B_1 xy + B_2 y + B_3 x^2 + B_5 x + B_6)\frac{\partial}{\partial x} + (B_1 y^2 + B_3 xy + B_4 y + B_7 x + B_8)\frac{\partial}{\partial y}$

$$\begin{aligned} \therefore G &= B_1\left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right) + B_2\left(y\frac{\partial}{\partial x}\right) + B_3\left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right) + B_4\left(y\frac{\partial}{\partial y}\right) \\ &+ B_5\left(x\frac{\partial}{\partial x}\right) + B_6\left(\frac{\partial}{\partial x}\right) + B_7\left(x\frac{\partial}{\partial y}\right) + B_8\left(\frac{\partial}{\partial y}\right) \end{aligned} \tag{4.43}$$

which is *eight* parameter symmetry. Any $m - parameter$ can be separated into $m - one$ parameter symmetry by choosing certain parameters for particular values. Initially it is set such that *one* parameter equal to *one* and other equal to *zero* in that order. Using

(4.43) it can generate an $eight - one$ parameter symmetry given by:

$G_1 = \frac{\partial}{\partial x}$

$G_2 = \frac{\partial}{\partial y}$

$G_3 = x\frac{\partial}{\partial x}$

$G_4 = y\frac{\partial}{\partial x}$

$G_5 = y\frac{\partial}{\partial y}$

$G_6 = x\frac{\partial}{\partial y}$

$G_7 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$

$$G_8 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{4.44}$$

$[G_1, G_3] = [G_1G_3] - [G_3G_1]$

$= \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) - x\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right)$

$= \frac{\partial}{\partial x} + x\frac{\partial^2}{\partial y\partial x} - x\frac{\partial^2}{\partial x\partial y}$

$= \frac{\partial}{\partial x}$

$= G_1$

$[G_1, G_6] = [G_1G_6] - [G_6G_1]$

$= \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) - x\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right)$

$= \frac{\partial}{\partial y} + x\frac{\partial^2}{\partial x\partial y} - x\frac{\partial^2}{\partial y\partial x}$

$= \frac{\partial}{\partial y}$

$= G_2$

$[G_1, G_7] = [G_1G_7] - [G_7G_1]$

$= \frac{\partial}{\partial x}\left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right) - \left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x}$

$= y\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial x\partial y} - xy\frac{\partial^2}{\partial x^2} - y^2\frac{\partial^2}{\partial y\partial x}$

$[G_2, G_4] = [G_2G_4] - [G_4G_2]$

$= \frac{\partial}{\partial y}\left(y\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)$

$= \frac{\partial}{\partial x} + y\frac{\partial^2}{\partial y\partial x} - y\frac{\partial^2}{\partial x\partial y}$

$$= \frac{\partial}{\partial x}$$

$$= G_1$$

$$[G_2, G_5] = [G_2 G_5] - [G_5 G_2]$$

$$= \frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\right) - y\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)$$

$$= \frac{\partial}{\partial y} + y\frac{\partial^2}{\partial y^2} - y\frac{\partial^2}{\partial y^2}$$

$$= \frac{\partial}{\partial y}$$

$$= G_2$$

$$[G_2, G_8] = [G_2 G_8] - [G_8 G_2]$$

$$= \frac{\partial}{\partial y}\left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right) - \left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}$$

$$= x^2\frac{\partial^2}{\partial y\partial x} + x\frac{\partial}{\partial y} + xy\frac{\partial^2}{\partial y^2} - x^2\frac{\partial^2}{\partial x\partial y} - xy\frac{\partial^2}{\partial y^2}$$

$$= x\frac{\partial}{\partial y}$$

$$= G_6$$

$$[G_3, G_4] = [G_3 G_4] - [G_4 G_3]$$

$$= x\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)$$

$$= yx\frac{\partial^2}{\partial x^2} - y\frac{\partial}{\partial x} - xy\frac{\partial^2}{\partial x^2}$$

$$= -y\frac{\partial}{\partial x}$$

$$= -G_4$$

$$[G_3, G_5] = [G_3 G_5] - [G_5 G_3]$$

$$= x\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial y}\right) - y\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial x}\right)$$

$$= yx\frac{\partial^2}{\partial y\partial x} - xy\frac{\partial^2}{\partial x\partial y}$$

$$= 0$$

$$[G_3, G_6] = [G_3 G_6] - [G_6 G_3]$$

$$= x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) - x\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial x}\right)$$

$$= x\frac{\partial}{\partial y} + x^2\frac{\partial^2}{\partial x\partial y} - x^2\frac{\partial^2}{\partial y\partial x}$$

$$= x\frac{\partial}{\partial y}$$

$$= G_6$$

$$[G_3, G_7] = [G_3 G_7] - [G_7 G_3]$$

$$= x\frac{\partial}{\partial x}\left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right) - \left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial x}\right)$$

$= xy\frac{\partial}{\partial x} + x^2y\frac{\partial^2}{\partial x^2} + y^2x\frac{\partial^2}{\partial x\partial y} - xy\frac{\partial}{\partial x} - x^2y\frac{\partial^2}{\partial x^2} - xy\frac{\partial}{\partial x} - x^2y\frac{\partial^2}{\partial x^2} - xy^2\frac{\partial^2}{\partial y\partial x}$

$= -xy\frac{\partial}{\partial x} - x^2y\frac{\partial^2}{\partial x^2}$

$[G_4, G_5] = [G_4G_5] - [G_5G_4]$

$= y\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial y}\right) - y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial x}\right)$

$= y^2\frac{\partial^2}{\partial x\partial y} - y\frac{\partial}{\partial x} - y^2\frac{\partial^2}{\partial y\partial x}$

$= -y\frac{\partial}{\partial x}$

$= -G_4$

$[G_4, G_6] = [G_4G_6] - [G_6G_4]$

$= y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}\right) - x\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial x}\right)$

$= y\frac{\partial}{\partial y} + xy\frac{\partial^2}{\partial x\partial y} - x\frac{\partial}{\partial x} - yx\frac{\partial^2}{\partial y\partial x}$

$= y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}$

$[G_4, G_8] = [G_4G_8] - [G_8G_4]$

$= y\frac{\partial}{\partial x}\left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right) - \left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial x}\right)$

$= 2xy\frac{\partial}{\partial x} + x^2y\frac{\partial^2}{\partial x^2} + y^2\frac{\partial}{\partial y} + xy^2\frac{\partial^2}{\partial x\partial y} - yx^2\frac{\partial^2}{\partial x^2} - xy\frac{\partial}{\partial x} - y^2x\frac{\partial^2}{\partial y\partial x}$

$= xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$

$= G_7$

$[G_5, G_7] = [G_5G_7] - [G_7G_5]$

$= y\frac{\partial}{\partial y}\left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right) - \left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial y}\right)$

$= xy\frac{\partial}{\partial x} + xy^2\frac{\partial^2}{\partial y\partial x} + 2y^2\frac{\partial}{\partial y} + y^3\frac{\partial^2}{\partial y^2} - y^2x\frac{\partial^2}{\partial x\partial y} - y^2\frac{\partial}{\partial y} - y^3\frac{\partial^2}{\partial y^2}$

$= xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$

$= G_7$

$[G_6, G_8] = [G_6G_8] - [G_8G_6]$

$= x\frac{\partial}{\partial y}\left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right) - \left(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y}\right)$

$= x^3\frac{\partial^2}{\partial y\partial x} + x^2\frac{\partial}{\partial y} + x^2y\frac{\partial^2}{\partial y^2} - x^2\frac{\partial}{\partial y} - x^3\frac{\partial^2}{\partial x\partial y} - x^2y\frac{\partial^2}{\partial y^2}$

$= 0$

Now after the above calculations, the non-zero Lie brackets are given as follows:

$[G_1, G_3] = G_1$

$[G_1, G_6] = G_2$

$[G_2, G_4] = G_1$

$[G_2, G_5] = G_2$

$[G_2, G_8] = G_6$

$[G_3, G_4] = -G_4$

$[G_3, G_6] = G_6$

$[G_4, G_5] = -G_4$

$[G_4, G_8] = G_7$

$[G_5, G_7] = G_7$

The process of finding the symmetries of ordinary differential equations is highly systematic. Thus let

$S_1 = \frac{\partial}{\partial x}$

$S_3 = x\frac{\partial}{\partial x}$

which are the Lie solvable algebra of the admitted $eight-one$ parameter symmetry (4.44).

By solving using prolongation:

$G^{[0]} = \omega\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y}$

$G^{[1]} = G^{[0]} + (\phi' - \omega' y')\frac{\partial}{\partial y'}$

$G^{[2]} = G^{[1]} + (\phi'' - 2\omega' y'' - \omega'' y')\frac{\partial}{\partial y''}$

$$\therefore G^{[3]} = G^{[2]} + (\phi''' - 3\omega' y''' - 3\omega'' y'' - \omega''' y')\frac{\partial}{\partial y'''} \tag{4.45}$$

Illustration of Differential Invariant (I)

Consider the third order prolongation for the operator:

$G = S = \omega\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y}$

If

$S_1 = \frac{\partial}{\partial x}$

it follows that :

$S_1^{[0]} = 1 \bullet \frac{\partial}{\partial x} + 0 \bullet \frac{\partial}{\partial y}$ (Since $\omega = 1$ and $\phi = 0$)

$S_1^{[0]} = 1 \bullet \frac{\partial}{\partial x} + 0$

$\therefore S_1^{[0]} = \frac{\partial}{\partial x}$

$S_1^{[1]} = S_1^{[0]} + (\phi' - \omega' y') \frac{\partial}{\partial y'}$

$S_1^{[1]} = S_1^{[0]} + (0 - 0 \bullet y') \frac{\partial}{\partial y'}$

$= S_1^{[0]} + 0 \bullet \frac{\partial}{\partial y'}$

$= S_1^{[0]}$

$\therefore S_1^{[1]} = \frac{\partial}{\partial x}$

$S_1^{[2]} = S_1^{[1]} + (\phi'' - 2\omega' y'' - \omega'' y') \frac{\partial}{\partial y''}$

$S_1^{[2]} = S_1^{[1]} + (0 - 2 \bullet 0 \bullet y'' - 0 \bullet y') \frac{\partial}{\partial y''}$

$= S_1^{[1]} + 0 \bullet \frac{\partial}{\partial y''}$

$= S_1^{[1]}$

$\therefore S_1^{[2]} = \frac{\partial}{\partial x}$

$S_1^{[3]} = S_1^{[2]} + (\phi''' - 3\omega' y''' - 3\omega'' y'' - \omega''' y') \frac{\partial}{\partial y'''}$

$S_1^{[3]} = S_1^{[2]} + (0 - 3 \bullet 0 \bullet y''' - 3 \bullet 0 \bullet y'' - 0 \bullet y') \frac{\partial}{\partial y'''}$

$= S_1^{[2]} + 0 \bullet \frac{\partial}{\partial y'''}$

$= S_1^{[2]}$

$= \frac{\partial}{\partial x}$

Hence

$$\therefore S_1^{[3]} = 1 \bullet \frac{\partial}{\partial x} + 0 \bullet \frac{\partial}{\partial y} \tag{4.46}$$

By solving for the characteristic:

$$\frac{dx}{1} = \frac{dy}{0} \tag{4.47}$$

$dy = 0$

and integrating yields the differential invariant:

$$y = U \tag{4.48}$$

where U is a constant, a function of x .

Illustration of Differential Invariants (II)

Consider the third order prolongation of the operator:

$G = S = \omega\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y}$

If

$S_3 = x\frac{\partial}{\partial x}$

it follows that :

$S_3^{[0]} = x\frac{\partial}{\partial x} + 0 \bullet \frac{\partial}{\partial y}$ (Since $\omega = x$ and $\phi = 0$)

$S_3^{[0]} = x\frac{\partial}{\partial x} + 0$

$\therefore S_3^{[0]} = x\frac{\partial}{\partial x}$

$S_3^{[1]} = S_3^{[0]} + (\phi' - \omega' y')\frac{\partial}{\partial y'}$

$S_3^{[1]} = S_3^{[0]} + (0 - 1 \bullet y')\frac{\partial}{\partial y'}$

$= S_3^{[0]} - y'\frac{\partial}{\partial y'}$

$\therefore S_3^{[1]} = x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'}$

$S_3^{[2]} = S_3^{[1]} + (\phi'' - 2\omega' y'' - \omega'' y')\frac{\partial}{\partial y''}$

$S_3^{[2]} = S_3^{[1]} + (0 - 2 \bullet 1 \bullet y'' - 0 \bullet y')\frac{\partial}{\partial y''}$

$= S_3^{[1]} + (0 - 2y'' - 0)\frac{\partial}{\partial y''}$

$= S_3^{[1]} - 2y''\frac{\partial}{\partial y''}$

$\therefore S_3^{[2]} = x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''}$

$S_3^{[3]} = S_3^{[2]} + (\phi''' - 3\omega' y''' - 3\omega'' y'' - \omega''' y')\frac{\partial}{\partial y'''}$

$S_3^{[3]} = S_3^{[2]} + (0 - 3 \bullet 1 \bullet y''' - 3 \bullet 0 \bullet y'' - 0 \bullet y')\frac{\partial}{\partial y'''}$

$= S_3^{[2]} + (0 - 3y''' - 0 - 0)\frac{\partial}{\partial y'''}$

$= S_3^{[2]} - 3y'''\frac{\partial}{\partial y'''}$

Hence

$$\therefore S_3^{[3]} = x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''} - 3y'''\frac{\partial}{\partial y'''} \tag{4.49}$$

By solving for the characteristics:

$$\frac{dy}{1} = \frac{dy'}{-y'} = \frac{dy''}{-2y''} = \frac{dy'''}{-3y'''} \tag{4.50}$$

Then integrate (4.50) to get the differential invariants as follows:

$(i)\frac{dy}{1} = \frac{dy'}{-y'}$

$dy = -\frac{dy'}{y'}$

$y = -\ln|y'| + \ln|C_1|$

$y = \ln|C_1| - \ln|y'|$

$$\therefore y = \ln|\frac{C_1}{y'}| \tag{4.51}$$

where C_1 is a constant.

$(ii)\frac{dy'}{-y'} = \frac{dy''}{-2y''}$

$\frac{dy'}{y'} = \frac{1}{2}\left(\frac{dy''}{y''}\right)$

$\ln|y'| = \frac{1}{2}\ln|y''| + \ln|C_2|$

$\Rightarrow \ln|y'| = \ln|y''|^{\frac{1}{2}} + \ln|C_2|$

$\ln|y'| = \ln|C_2||y''|^{\frac{1}{2}}$

$\Rightarrow y' = C_2(y'')^{\frac{1}{2}}$

$C_2 = \frac{y'}{(y'')^{\frac{1}{2}}}$

$= \frac{(y')^2}{y''}$

If

$\frac{1}{C_2} = \frac{y''}{(y')^2}$

Let

$t_1 = \frac{1}{C_2}$

$$\therefore t_1 = \frac{y''}{(y')^2} \tag{4.52}$$

where C_2 and t_1 are constants.

$(iii)\frac{dy'}{-y'} = \frac{dy'''}{-3y'''}$

$\frac{dy'}{y'} = \frac{dy'''}{3y'''}$

$\frac{dy'}{y'} = \frac{1}{3}\left(\frac{dy'''}{y'''}\right)$

$\ln|y'| = \frac{1}{3}\ln|y'''| + \ln|C_3|$

$\Rightarrow \ln|y'| = \ln|y'''|^{\frac{1}{3}} + \ln|C_3|$

$\ln|y'| = ln|C_3||y'''|^{\frac{1}{3}}$

$\Rightarrow y' = C_3(y''')^{\frac{1}{3}}$

$C_3 = \frac{y'}{(y''')^{\frac{1}{3}}}$
$= \frac{(y')^3}{y'''}$

If

$\frac{1}{C_3} = \frac{y'''}{(y')^3}$

Let

$t_2 = \frac{1}{C_3}$

$$\therefore t_2 = \frac{y'''}{(y')^3} \tag{4.53}$$

where C_3 and t_2 are constants.

$(iv)\frac{dy''}{-2y''} = \frac{dy'''}{-3y'''}$
$\frac{dy''}{2y''} = \frac{dy'''}{3y'''}$
$\frac{1}{2}\left(\frac{dy''}{y''}\right) = \frac{1}{3}\left(\frac{dy'''}{y'''}\right)$
$\frac{1}{2}\ln|y''| = \frac{1}{3}\ln|y'''| + \ln|C_4|$

$\Rightarrow \ln|y''|^{\frac{1}{2}} = \ln|y'''|^{\frac{1}{3}} + \ln|C_4|$

$\ln|y''|^{\frac{1}{2}} = \ln|C_4||y'''|^{\frac{1}{3}}$

$\Rightarrow (y'')^{\frac{1}{2}} = C_4(y''')^{\frac{1}{3}}$

$C_4 = \frac{(y'')^{\frac{1}{2}}}{(y''')^{\frac{1}{3}}}$
$= \frac{(y'')^3}{(y''')^2}$

If

$\frac{1}{C_4} = \frac{(y''')^2}{(y'')^3}$

Let

$t_3 = \frac{1}{C_4}$

$$\therefore t_3 = \frac{(y''')^2}{(y'')^3} \tag{4.54}$$

where C_4 and t_3 are constants.

By taking (4.52) then:

$t_1 = \frac{y''}{(y')^2}$

Let

$t_1 = V$

then (4.52) becomes :

$$\therefore V = \frac{y''}{(y')^2} \tag{4.55}$$

Now reducing (4.2) to first order ODE(Dresner, 1999) yields:

$\frac{dV}{dy} = \frac{D_x(V)}{D_x(y)}$

$= \frac{D_x\left(\frac{y''}{y'^2}\right)}{D_x(y)}$

$= \frac{y'''(y')^2}{(y')^5} - \frac{2y'y''y''}{(y')^5}$

$= \frac{y'''}{(y')^3} - \frac{2y''y''}{(y')^4}$

$= \frac{y'(y'')^4(y)^{-4}}{(y')^3} - \frac{2y''y''}{(y')^2(y')^2}$

$\frac{dV}{dy} = \frac{y'(y'')^4(y)^{-4}}{(y')^3} - \frac{2(y'')(y'')}{(y')^2(y')^2}$

$\frac{dV}{dy} = (y'')^4(y')^{-2}(y)^{-4} - \frac{2(y'')(y'')}{(y')^2(y')^2}$

From (4.48) and (4.55) through substitution leads to :

$\frac{dV}{dy} = (y'')^4(y')^{-2}(y)^{-4} - 2V^2$

$$\therefore \frac{dV}{dy} + 2V^2 = (y'')^4(y')^{-2}(y)^{-4} \tag{4.56}$$

Then (4.56) is of the form:

$$\frac{dV}{dy} + P(y)V = Q(y) \tag{4.57}$$

implying that it has been managed to reduce a third order equation (4.2) to a simple first order linear equation (4.56) that is easily solvable by other known simpler methods. If

$P(y) = 2V$

and

$Q(y) = (y'')^4(y')^{-2}(y)^{-4}$

Then (4.2) reduces to (4.57) which can be easily integrated using integrating factors given by:

$I(y)$

Thus

$$I(y) = e^{\int P(y)dy} \tag{4.58}$$

$I(y) = e^{\int 2V dy}$

$I(y) = e^{2\int V dy}$

$= e^{2\int \frac{y''}{(y')^2} dy}$

$= e^{2\ln|y'|^2 + C}$

$= e^{\ln|y'|^4} \bullet e^C$

$= Me^{\ln|y'|^4}$ (If $e^C = M$)

$= e^{\ln|y'|^4}$ (Since $C = 0,\ M = 1$) then

$I(y) = e^{\ln|y'|^4}$

$= (y')^4$

where C and M are constants. From the form :

$$V = \frac{1}{I(y)} \int (y')^4 Q(y) dy \tag{4.59}$$

Then it follows that :

$V = \frac{1}{(y')^4} \int (y')^4 [(y'')^4 (y')^{-2} (y)^{-4}] dy$

whose simplification leads to :

$$\therefore V = \frac{1}{(y')^4} \int (y'')^4 (y')^2 (y)^{-4} dy \tag{4.60}$$

which completes the process of integration hence (4.60) is the simple first order form of the required mathematical solution of the special type wave equation (4.2) namely :

$$y''' - y'\left(\frac{y''}{y}\right)^4 = 0$$

4.3 The General Solution of Nonlinear Wave Equation of Third Order

Now consider the general form of the given wave equation (4.2) as:

$$U''' - U'\left(\frac{U''}{U}\right)^4 = 0 \tag{4.61}$$

Again by taking (4.48) and (4.52) such that:

$y = U$

and

$t_1 = \frac{U''}{(U')^2}$

Let $t_1 = V$ then (4.52) becomes :

$$\therefore V = \frac{U''}{(U')^2} \tag{4.62}$$

Now reducing (4.61) to first order ODE(Dresner, 1999) yields:

$\frac{dV}{dU} = \frac{D_x(V)}{D_x(U)}$

$= \frac{D_x\left(\frac{U''}{U'^2}\right)}{D_x(U)}$

$= \frac{U'''(U')^2}{(U')^5} - \frac{2U'U''U''}{(U')^5}$

$= \frac{U'''}{(U')^3} - \frac{2U''U''}{(U')^4}$

$= \frac{U'(U'')^4(U)^{-4}}{(U')^3} - \frac{2U''U''}{(U')^2(U')^2}$

$\frac{dV}{dU} = \frac{U'(U'')^4(U)^{-4}}{(U')^3} - \frac{2(U'')(U'')}{(U')^2(U')^2}$

$\frac{dV}{dU} = (U'')^4(U')^{-2}(U)^{-4} - 2(\frac{U''}{(U')^2})(\frac{U''}{(U')^2})$

From (4.48) and (4.62) through substitution leads to :

$\frac{dV}{dU} = (U'')^4(U')^{-2}(U)^{-4} - 2V^2$

$$\therefore \frac{dV}{dU} + 2V^2 = (U'')^4(U')^{-2}(U)^{-4} \tag{4.63}$$

Then (4.63) is of the form:

$$\frac{dV}{dU} + P(U)V = Q(U) \tag{4.64}$$

implying that it has been managed to reduce a third order equation (4.61) to a simple first order linear equation (4.63) that is easily solvable by other known simpler methods. If

$P(U) = 2V$

and

$Q(U) = (U'')^4(U')^{-2}(U)^{-4}$

Then (4.61) reduces to (4.64) which can be easily integrated using integrating factors given by:

$I(U)$

Thus

$$I(U) = e^{\int P(U)dU} \tag{4.65}$$

$I(U) = e^{\int 2V dU}$

$I(U) = e^{2\int V dU}$

$= e^{2\int \frac{U''}{(U')^2} dU}$

$= e^{2\ln|U'|^2 + C}$

$= e^{\ln|U'|^4} \bullet e^C$

$= Me^{\ln|U'|^4}$ (If $e^C = M$)

$= e^{\ln|U'|^4}$ (Since $C = 0$, $M = 1$) then

$\therefore I(U) = e^{\ln|U'|^4}$

$= (U')^4$

where C and M are constants. From the form :

$$V = \frac{1}{I(U)} \int (U')^4 Q(U) dU \tag{4.66}$$

Then it follows that :

$V = \frac{1}{(U')^4} \int (U')^4[(U'')^4(U')^{-2}(U)^{-4}]dU$

whose simplification leads to :

$$\therefore V = \frac{1}{(U')^4} \int (U'')^4(U')^2(U)^{-4}dU \tag{4.67}$$

hence (4.67) is a simple first order form of the required general solution of the special type wave equation (4.61) namely :

$$U''' - U' \left(\frac{U''}{U}\right)^4 = 0$$

CHAPTER 5

SUMMARY,CONCLUSION AND RECOMMENDATIONS

5.1 Summary

The Mathematical Solution of Nonlinear Wave Equation of Third Order

This study looked at a special case of a wave equation, that is, a third order first degree nonlinear, nonhomogeneous ODE of fourth degree in second derivative of the form (4.2) namely :

$$y''' - y'\left(\frac{y''}{y}\right)^4 = 0 \tag{5.1}$$

and whose mathematical solution is:

$$V = \frac{1}{(y')^4}\int (y'')^4(y')^2(y)^{-4}dy \tag{5.2}$$

To obtain this solution (5.2), the method of Lie symmetry was employed because unlike other numerical methods which give solutions that are approximations, it yields exact solutions to given wave equations. In the Lie symmetry analysis while manipulating (5.1), the following were applied: laws of indices together with the removal of the fractions in order to get the transformation equation, the third extension of $G^{[3]}$, substitutions, expansions, simplifications, the partial derivatives, integration, identities, infinitesimal generators, one-parameter symmetry, non-zero Lie brackets, Lie solvable algebra, third order prolongation of the operator, solving for the characteristics, differential invariants and the reduction of third order to first order ODE. The simple first order ODE is easily solvable using other available methods like integration.

The General Solution of Nonlinear Wave Equation of Third Order

In this research work, after considering (5.1), which is a general wave equation similar to the special wave equation that was manipulated to obtain the mathematical solution

(5.2), then the general wave equation was of the form (5.3):

$$U''' - U' \left(\frac{U''}{U} \right)^4 = 0 \tag{5.3}$$

where U is a function of x and yielded a general solution of the form (5.4):

$$V = \frac{1}{(U')^4} \int (U'')^4 (U')^2 (U)^{-4} dU \tag{5.4}$$

The Lie symmetry analysis was used to obtain the general solution since it leads to an exact solution. To get this general solution, the study employed the following: the Lie groups of transformations, infinitesimal transformations, one-parameter Lie group, infinitesimal generators, Lie algebras, prolongations, variation symmetries, differential invariants, Lie point symmetries, integrating factor and reduction of third order to first order which is easier to solve using other methods.

5.2 Conclusion

The wave phenomena has created the world into the so called electrical brain waves! Wave phenomena affect lives in many ways such that locomotion to surfers is furnished by water waves. Many forms of nonlinear ordinary differential equations occur in the analysis of problems found in physics, engineering, chemistry and biology. The main focus was to determine a mathematical solution of the nonlinear wave equation of third order using Lie symmetry analysis. The aim was not to study a particular solution but to acquaint you with Lie symmetry procedures that are common to these types of problems. In real life situations, the occurrence of earthquake tremors and sea or ocean waves in most cases often do great damage to humanity and properties. For example, the shock waves from the jet planes rattle and break window panes. The wave theory, the nature of the waves and the industrial application of their properties, are growing at an explosive rate. Scientists are supposed to put some mechanisms in place so as to minimize large scale damages. They need to have knowledge about water waves in terms of their nature in order to tell the time they are likely to happen and the extent of their destruction. The important parameters involving waves are the amplitude (maximum displacement), velocity (speed), frequency, wavelength and the periodic time. The decrease in velocity as waves approach a shallow region is consistent with the behavior of ocean waves. By taking the peaks and troughs of water waves as points sitting over deeper and shallower parts respectively, the wave velocity at the peaks is higher than that at the troughs. This causes the crests to break into a splash as they approach the shallows of the continental shelf. In sensitive sonar equipment, which use high-frequency sound waves, permits the fast, economical, and accurate charting of the floor of the oceans, and also detects the presence of submarines and schools of fish. Internal waves which are large have been found far below the surface of the sea. The exploration of the seas and oceans is becoming important because as the worlds population grows, man must turn his attention to the waters that cover four-fifths of the earths surface. Ground waves whether created by explosives or earthquakes, are used in geology and geophysics for oil exploration and for investigating properties of the earths core. In chemistry, waves determine the crystalline structure, and in physics they explore atoms and subatomic particles. For a radio operator or hi-fi enthusiast and you would like design computer devices or television equipment or work as a theoretical or experimental physical scientist, you need to understand the wave theory. Since the manipulation of equation (1.48) yielded a mathematical solution then the first objective was achieved. This implies that a solution exists. This solution can be worked out using other simpler known methods like integration in order to get a particular solution.

In the second objective, the interest was to have a general solution of the general form that could be used by other mathematicians, engineers and researchers in science to solve specified wave equations. This forms the basis of future predictions which aim at saving human loss of life and properties. For example once the range of the amplitude of a particular wave across the ocean has been calculated, a decision can be made to clear the vicinity including human evacuation in order to minimize damages. Again, if the velocity range of a given wave is established and its strength known then the damages such a wave could cause may be estimated and safety measures taken promptly. Also from the velocity of a wave, time taken to reach a particular point and area can be evaluated and the required mechanisms put in place early enough before such a wave reaches such a point to cause disaster. Hence the objective was highly achieved.

5.3 Recommendations

It is wished that further research may attempt the solution of similar wave equation but of fourth or higher order nonlinear ordinary differential equation since there are no known researches that have been carried out lately. There is also a need for other researchers in the field of technology to develop a computer package which can be used to determine solutions to similar nonlinear ordinary differential equations in future because the current methods are very long and tedious.

APPENDICES

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APPENDICES

APPENDIX A: PLAGIARISM REPORT

PUBLICATION OF TEXT BOOKS

APPENDIX B: NOTIFICATION OF PUBLICATION ONE

PUBLICATION OF FIRST PAPER

APPENDIX C: NOTIFICATION OF PUBLICATION TWO

PUBLICATION OF SECOND PAPER

APPENDIX D: NOTIFICATION OF PUBLICATION THREE