NORM ESTIMATES FOR NORM-ATTAINABLE DERIVATIONS IN BANACH ALGEBRAS

JANES NYABONYI ZACHARY

BACHELORS OF EDUCATION SCIENCE (EGERTON UNIVERSITY)

A THESIS SUBMITTED TO BOARD OF POSTGRADUATE STUDIES IN PARTIAL FULFILLMENT FOR THE REQUIREMENT OF THE AWARD OF THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS IN THE SCHOOL OF PURE AND APPLIED SCIENCES, DEPARTMENT OF MATHEMATICS AND ACTUARIAL SCIENCE, KISII UNIVERSITY

DECEMBER, 2022

DECLARATION AND RECOMMENDATION.

STUDENT

Signature.......................... Date....11/11/22..............................

JANES NYABONYI ZACHARY

Reg.No. MPS17/00004/19

Department of Mathematics and Actuarial Science

Kisii University

P.O Box 408-40200, Kisii

This thesis submission for marking was done with our consent as the supervisors.

SUPERVISORS

Signature......................................

Date..........14/11/2022...........

Prof. Benard Okelo

Department of Pure and Applied Mathematics

Jaramogi Oginga Odinga University of Science and Technology, Kenya

P.O BOX 30197-0010, Bondo

Signature......................................

Date.....18/11/22.....................

Dr. Robert Obogi

Department of Mathematics and Actuarial Science

Kisii University

P.O Box 408-40200, Kisii

PLAGIARISM DECLARATION

DECLARATION BY THE CANDIDATE.

I have fully understood Kisii University rules and regulations by reading through the documents concerning academic plagiarism.

The thesis entails my original work and wherever somebody's work has been used (whether from a printed source, the internet or any other source) reference and due acknowledgement have been made in regard to the Kisii University's policy.

1) I'm aware that I must legally own the work.
2) I'm aware that in case of dishonesty my work can be allotted a fail grade ("F").
3) I'm also aware that in case of plagiarism I may be suspended or expelled from the University.

JANES NYABONYI ZACHARY

MPS17/00004/19

Sign............... Date..11/11/22..........

DECLARATION BY SUPERVISORS

1) We hereby confirm that the document has been put through plagiarism detection process.
2) The thesis entails not more than 20% of plagiarized information.
3) Therefore we accord it for examination.

1. Name: **Prof. Benard Okelo**

Signature.................................. Affiliation: **JOOUST**

Date........14/11/2022..............

2. Name: **Dr. Robert Obogi**

Signature.................................. Affiliation: **Kisii University**

Date....18/11/22..................

DECLARATION OF THE NUMBER OF WORDS

Declaration by the candidate

I confirm that word length of:

1) The thesis is 22,201 words.
2) The bibliography, 1000 words.
3) The appendices are 91 words.

I also declare the electronic version is identical to the final, hardbound copy of the thesis and corresponds with those on which the examiners based their recommendation for the award of the degree

Candidate Name: Janes Nyabonyi Zachary **Reg. No: MPS17/00004/19**

Signed: .... Date 11/11/22

(Candidate)

I confirm that the thesis submitted by the above named candidate complies with the relevant word length in the school of Postgraduate and Commission of University Education regulations for Master's Degree.

Prof. Benard Okelo, PhD .... Date 14/11/2022
Sign

Dr. Robert Obogi, PhD .... Date 18/11/22
Sign

COPY RIGHT

DEDICATION

To my hubby Elias, daughters; Mirriam, Ruth and Deborah.

ACKNOWLEDGEMENT

I am grateful to God for good health and opportunity to education. To my supervisors Prof. Benard Okelo and Dr. Robert Obogi I say thank you for your constant engagement in supp orting my masters research. I recognize their commitment towards helping me to come up-with this document. I also give thanks to Mr. Asamba, Mr. Sabasi and Mr. Tinega for your encouragement and readiness to guide me when I consulted you. I thank my family for ins-piring me and giving me moral, spiritual, emotional and financial support . To teacher Mrs. Omache, you have been a good affiliate and guider.

Index of Notations

ABSTRACT

The investigation of Norm-attainability in Hilbert spaces and derivations has been done for quite long. Norm-attainability conditions for elementary operators such basic elementary operator and Jordan elementary operator has been done and results obtained . But, norm-attainable conditions for derivations in Banach-algebras and norm-estimates that is upper and lower norm-estimates for derivations in Banach algebras has not been done. Objectively this study will: Establish norm-attainability conditions for derivations in Banach-algebras, determine upper and lower norm estimates for norm-attainable derivations in Banach algebras. The research methods used involvees use inequalities well known such as Cauchy-Schwarz, Triangle,H¨olders and Bessel's inequality. Technically, Direct sum decomposition, Polar decomposition and Tensor product methods were used. Results obtained from this study will be useful in quantum mechanics and in integration.

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

1.1 Mathematical background

Banach algebras are key in several studies in mathematics and the advancement in both trivial and non-trivial cases in mathematics and quan-tum mechanics. The norm of a derivation was first introduced by Stampfli [87], who determined the inner derivation $\delta_{T0} : A_0 \to T_0A_0 - A_0T_0$ which acts on Banach algebra $B(H)$ on Hilbert space H. Further, $\| \delta_{T0} \| = \inf 2 \| T_0 - \lambda I_0 \|$ for every complex λ was shown. For a normal T , then $\|\delta_{T0} \|$ can be expressed as the geometry of the spectrum of T_0. Johnson [37] established method which apply to a uniform convex spaces with a large class, that is the formula $\|\delta_T \|$ is false in l^r and $L^r(0, 1)$ $1 < r < \infty, r \neq 2$. Johnson [36] found that $B(H)$ derivation is a map $\delta : B(H) \to B(H)$ with $\delta(P S) = P \delta(S) + \delta(P)S$ $P, S \in B(H)$. Such derivations are necessarily continuous and if $S \in B(H)$ then $\delta_S(P) = P S - SP$ is a derivation in $B(H)$.

Gajendragadka [27] was concerned with computation of norm of derivation and Von-Neumann algebra. Specifically when the Von-Neumann algebra act on separable Hilbert space *H*, $K \in$ U was proved then δ_K a derivative induced by *K*, then $\|\delta_K |U\| = 2 \inf \|K - M\|$, *M* in the centre U. Therefore, Anderson [5] in his investigation on normal derivation the operators $A, C \in B(H)$ were proved that *A* is normal if *AC* commutes, for every $Y \in B(H)$, $\|\delta_A(Y) + C\| \geq \|C\|$. Therefore, the inequality showed that the kernel and the range δ_A is orthogonal to δ_A which is commutation of $\{A\}'$ of *A*. Kyle [38] examined the numerical-range of in-ner derivation and the element generating the relationship between them. Kyle [39] studied on norms of inner derivations and used their properties and concluded that a C^*-algebra is a closed sub-set of entire derivation(s) which forms the inner derivations set and obtained the result which was a converse by Stampfli [87].

Charles and Steve [16] answered the question when $X = K$ by struc-ture characterization of compact derivations of C^*-algebras. Moreover, the structure of weak compact derivations of C^*-algebras was determined and as immediate corollaries of these results, conditions that were nec-essary and sufficient were obtained so that C^*-algebras admits non-zero compact or weakly compact derivation. Stampfli [88] studied operators on Hilbert space and their properties inducing a derivation whose closure is self-adjoint after range such operators are termed *D*-symmetric and then characterized compact *D*-symmetric operators. Further, considera-tion was given to normal derivations and then presented an irreducible, not essential normal *D*-symmetric operators as an example. Erik [23] established that any C^*-algebra *F* on a Hilbert space *H* with cyclic vec-tor whose derivative property δ of *F* into $B(H)$ an operator *y* existed in

$B(H): \forall f \in F, \delta(f) = [y, f] = yf - fy.$

Mecheri [53] established that $T(X)$ is linear for a m-linear derivation and hence, the topology of Von-Neumann algebra X of type I is automatically continuous in measure with center m and the semi-finite trace τ which is normal is faithful. Therefore, $K(X)$ is the algebra of all τ-measurable operators affiliated with X. Mathieu [48] proved that for bounded derivations that are non-zero then the product of two prime C^*-algebras are bounded. In Volker [92] two automatic continuity problems for deriva-tions on commutating Banach algebras were discussed : (a) Derivation on a commutative algebra is mapped onto the radical, and: (b) Banach algebras are continuous on semiprime derivations. It was proved that

(b) implies (a). Furthermore, (b) proved that for special cases Banach algebras are reduced to a small class and also similar results were given on epimorphisms. In fact, it was shown that semisimple Banach algebras were characterized with no topologically nilpotent element other than zero being among the commutative Banach algebras; known examples of discontinuous derivations on commutating Banach algebras depended ma-jorly on the existing nontrivial nilpotent elements which was on a general-ized derivation of semiprime Banach algebra and that nilpotent elements are continuous on a commutative Banach algebra without nontrivial.

Bresar, Zalar [13] showed that a Jordan $*$-derivation is the map $\delta_a(x) = ax - x^*a$ for fixed $a \in$ U; hence, the derivation is inner and the follow-ing are the results obtained. Douglas [21] continued the study of $W_s(N)$ which was considerably more amenable where Archbold [1] defined the smallest numbers to be $[0, \infty]$ and introduced two constants $W(N)$ and $W_x(N)$: $d(n, Q(N)) \leq W(N) \| D(n, N) \| \ \forall n \in N$ and $d(n, Q(N)) \leq$

$W_s(N)\|D(n, N)\|$ $\forall n = n^* \in N$. Dutta, Nath, Kalita [22] showed that if α_1 and α_2 are δ-derivation and δ'-derivation on (T, γ) and (T', γ') and an arbitrary element $n = \sum_1 y_1 \otimes x_1$ of $(T, \gamma) \otimes_\rho (T', \gamma')$, then a derivation D on $\delta \otimes \delta'$ exists in $(T, \gamma) \otimes_\rho (T', \gamma')$ satisfy $\alpha(n) = \sum_1 [(\alpha_1 y_1) \otimes x_1 + y_1 \otimes (\alpha_2 x_1)]$ in which many enlightening properties were possessed. Furthermore, the validity of the results was investigated on $\|\alpha\| = \|\alpha_1\| + \|\alpha_2\|$ and $sp(\alpha) = sp(\alpha_1) + sp(\alpha_1)$.

Rajendra, Kalyan [77] showed that for the nth order commutator

$[[[k(B), Y], Y], ..., Y]$ a formula was obtained in terms of the Frechet derivatives $S^m k(B)$ in which the formula illustrated was used to obtain bounds for norms of a generalized commutator $k(B)Y - Y k(B)$ and their higher order analogues. In Joel [35] the numerical range of 2 x 2 matrices was determined, the convex of the numerical range for any Hilbert space operator was established by Toeplitz-Hausdorff theorem and relation of numerical-range to that of spectrum was discussed. Further, closure of the numerical-range is contained in the spectrum, the intersection of closures of the numerical-range of all operators were asserted by Hildebrandt's theorem that are similar to operator D was given precisely and discussed the convex hull of the spectrum of D. Considering results on special cases Blanco, Boumazgour, Ransford [12] established that $\|PXQ + QXP\| \geq \|P\|\|Q\|$.

Chi-Kwong [17] established that for a n x n matrix X, the numerical range $W(X)$ has many properties which can be used to locate eigenval-ues, to obtain norm bounds algebraic and analytic properties were de-duced which will help in finding the dilations of simple structure. The numerical radius of Y defined as $\omega(Y) = max_{\mu \in W(Y)}|\mu|$ and $\tilde{\omega}(Y) = min_{\mu \in W(Y)}|\mu|$ is the distance of $W(X)$ to the origin which is related to numerical range. $\omega(X)$ and $\tilde{\omega}(X)$ are useful quantities in studying convergence, stability, perturbation and approximation problems. Let the linear operators X_i and Y_i, $1 \leq i \leq n$ act on separable Hilbert space H. Hong-Ke, Yue-qing [33] proved that $\sup\{\|\sum_{i=1}^{n} R_i Y S_i\| : Y \in B(H), \|Y\| \leq 1\} = \sup\{\|\sum_{i=1}^{n} R_i T S_i\| : UU^* = T^*U = I, U \in B(H)\}$. Therefore, there exists an operator Y_k which proved that $\|Y_k\| = 1$ im-plying $\|\sum_{i=1}^{n} R_i Y_k S_i\| = \sup\{\|\sum_{i=1}^{n} R_i Y S_i\| : Y \in B(H), \|Y\| \leq 1\}$ only if there exists a unitary $U_0 \in B(H)$ so that $\|\sum_{i=1}^{n} R_i U_0 S_i\| = \sup\{\|\sum_{i=1}^{n} R_i Y S_i\| : Y \in B(H), \|Y\| \leq 1\}$.

Nyamwala and Agure [54] proved that $\|AXM + MXM\| = 2\|A\|\|M\|$ and in this study it was shown that $\|A\|\|M\| \leq \|AXM+AXM\| \leq 2\|A\|\|M\|$. In Nyamwala [55] the symmetry of a multiplication operator norm which is two-sided was calculated as $T_{P\,Q_k} X = PXQ_k + Q_k XP$ defined on a C^*-algebra $C^*P, Q_k, 1$ generated by P and Q_k for an idempotent X related to P and Q_k. In addition, Okelo, Agure and Ambogo [61] established the Jordan-elementary operator norm $U_{M,N} : B(H) \rightarrow B(H)$ given as $U_{M,N} = MYN + NYM$, $\forall Y \in B(H)$ and M, N in $B(H)$, showing that $\|U_{M,N}\| \geq \|M\|\|N\|$ and then characterized the norm-attainable operators using this norm. Okelo [68] investigated that ideals of norm-attainable elements implemented by inner derivations of a C^*-algebra has relation to primitive ideals. Since there is a relationship between the constants $A(\xi)$ and $A_s\xi$ ideals of C^*-algebras and ideals that are primitive then related results were given.

Okelo, Agure and Oleche [66] gave results on necessary and sufficient conditions for norm-attainable operators also studied norm-attainable op-erators and generalized derivations. Okelo [65] extended the work by presenting new results on conditions that are sufficient and necessary for norm-attainable operators on Hilbert space, elementary operator and gen-eralized derivation was established. Further, Okelo [65] established that a unit vector exists $\lambda \in H$, $\|\lambda\| = 1$ so that $\|S\lambda\| = \|S\|$ with $S\lambda, \lambda = \eta$. Hoger [32] showed that every Jordan derivation of the trivial extension of A by M, under some conditions, is the summation of the derivative and anti-derivative. Okelo, Ongati, Obogi [62] studied norm-attainable operators that are convergent and established projective tensor norm via norm-attainable operators.

Wickstead [94] showed that if atomic Banach lattice Z having a norm order that is continuous, $X, Y \in T^r$ and $M_{X,Y}$ are operators on $T^r(Z)$ given as $M_{X,Y}(A) = XAY$, then $\|M_{X,Y}\|_r = \|X\|_r\|Y\|_r$ with no real $\beta > 0$ hence $\|M_{X,Y}\|_r = \beta\|X\|_r\|Y\|_r$. Okelo [72] outlined the theory of self-adjoint and norm-attainable operators then presented norms of operators in Hilbert spaces. Sayed, Madjid, Hamid [80] proved that for a linear map

$\Delta : U \to U$, $\Delta(XY) = \Delta(X)Y + \Delta X(Y)$ for each $X, Y \in U$ is a derivation, then any two derivations Δ and Δ' on a C^*-algebra U exists a derivation $\delta \in U$ such that $\Delta\Delta' = \delta^2$ if and only if either $\Delta' = 0$ or $\Delta = f\Delta'$ for any $f \in \mathbb{C}$. Clifford [18] studied hypercyclic generalized derivations acting on separable ideals of operators then identified concrete examples and established some conditions that are necessary and sufficient for their hypercyclicity. Particular Banach algebras acted on by the dynamics of elementary operators were considered.

Oyake, Okelo and Ongati [74] characterized inner derivations in Banach algebra and investigated inner derivation properties that are implemented by norm-attainable operators such as measurability, normality continu-ity, linearity, trace and spectra of inducing operator and determined the norms. The result showed that the derivations admitted tensor norms of operators. Kinyanjui [42] characterized norm-attainable elementary op-erator and showed if operators M, P and $\delta_{M,P}$ be norm-attainable, then $\delta_{M,P}$ is normally represented. In Okelo and Aminer [67] norm inequali-ties of new matrices that are norm-attainable operators, were presented as well as mapping on matrices were characterized. Okelo and Aminer [67] completely characterized norms that are bounded, gave the extension of orthogonality via norm-convergence in $NA(H)$-classes. Okelo [64] consid-ered orthogonal and norm-attainable of operators in Banach spaces, gave in details the characterization and generalizations of norm-attainability and orthogonality. The conditions that are sufficient and necessary for norm-attainable operations on a Hilbert space, result on kernel of elemen-tary operators and the orthogonal range when done by norm-attainable operators in Banach-spaces were given.

Odero, Agure, Nyamwala [56] showed that the mapping $\Delta_{Q'R'} : B(H) \to B(H)$ is a generalized derivation of two operators that are bounded $Q', R' \in B(H)$ induced by Q' and R' were defined by $\Delta_{Q',R'}(Y) = Q'Y - YR'$, therefore, the norm $\|\Delta_{Q',R'}\| = \|Q'\| + \|R'\|$ for all $Q', R' \in B(H)$ was given. Okelo and Mogotu [59] gave norms of commutators of normal op-erators for generalized inequalities and established the commutations of derivation for orthogonality and norm inequalities. Okelo [60] character-ized norm-attainable classes in terms of orthogonality by giving norm-attainability conditions that were necessary and sufficient for Hilbert space operators first and the orthogonality result on the kernel and range of norm-attainable classes in elementary operators, implemented by op-erators that are norm-attainable were given. Okelo [63] gave conditions for linear functionals in Banach spaces for norm-attainable operators, el-ementary operators and non-power operators on H and also for power operators a new notion of norm-attainability was given and then charac-terized norm-attainable operators in normed spaces.

Abolfazl [2] determined the norm of inner Jordan $*$-derivations $\delta_S : T \to ST - T^*S$ that act on the Banach algebra $B(H)$. It was shown that $\|\delta_S\| \geq 2 \sup_{\lambda \in W_0(S)} |\Im\lambda|$ in which $W_0(S)$ is the maximal numerical range of operator S. Gyan [28] obtained precisely when zero belongs to maxi-mal numerical range of composition operators on H and then character-ized the norm-attainability of derivations on $B(H)$. In Okelo [71] norm-attainability for hyponormal operators that are compact were character-ized, sufficient conditions for a compact hyponormal operator that is lin-ear and bounded on an infinite dimension for a complex Hilbert space to be norm attainable were given. Further, the structure and other proper-ties of compact hyponormal operators when they are self-adjoint, normal and norm attainable with their commutators were discussed in general.

Lumer [46] obtained a sharp estimate not only from $|sp(R)|$ = spectral radius of R but also $|sp(R)|$ in terms of $\sup(|X(R)|, |X(R^n)|^{1/n})$, for an even integer n which is positive. These are

$$|sp(R)| \leq \sqrt{3} \sup(|X(R)|, |X(R^2)|^{1/2})$$

$$|sp(R)| \leq \sigma_n \sup(|X(R)|, |X(R^n)|^{1/n}),\ n = 4, 6, 8..., \text{ where } \sigma_n = \sqrt{7},$$

generally, σ_n can be calculated as a polynomial root which depend on

n. The question about the constants was answered completely for an estimate $\|R\| \le c_1|X(R)| + c_2|X(R^2)|^{1/2}$ which was expressed as

sup $\alpha(|X(R)|, \beta|X(R^2)|^{1/2})$ and then compared the estimates. Fur-ther, the aspects of general problem were discussed and then gave ap-plications by introducing an invariant $\delta(C)$ defined for all unital Banach algebra C.

Briggs [14] studied the algebra of functions that are continuous on [0, 1] that are $\|.\|_W$-approximate polynomial; that is point-wise functions of limit of $\|.\|_W$-Cauchy polynomial sequence. Let $C^1(W)$ be the algebra of all such functions, for comparison purposes two other algebras of functions were defined. If $W \in C[0, 1]$ let $L(W)$ be the zero set of W and $C^1_{(W)}$ be the subalgebra of $C[0, 1]$ that consist p such that $p'(x)$ existed for every $x \in [0' 1] \setminus L(W)$, the function Wp' is continuous on [0, 1] since $(Wp')(x) = 0$ if $x \in L(W)$, $(Wp')(x) = W(x)p'(x)$ if $x \in [0, 1] \setminus L(W)$ and let the subalgebra be AC_W of $C^1(W)$ which consist functions that are absolutely continuous.

Archbold [1] investigated whether the simple triangle inequality $\|T(a, A)\| \le 2t(a, Z)$ if applied holds. $D(A)$ was defined to be a minimum value D in $[0, \infty]$ so that $t(a, Z) \le D\|T(a, A)\|$. The behaviour of D in ideals and quotients were discussed which proved that $D_S(A) \le 1$ for a weakly central C^*-algebra A and considered a class of n-homogeneous C^*-algebras that are special. D and D_S was investigated and approximated finite-dimension $(AF)C^*$-algebra in that context and an example was given to show certain estimates. Shlomo [85] showed that for a certain Von-Neumann algebra U, a constant F existed such that dist(T, U) $\le F \sup_{P \in lat\mathrm{U}} \|P^{\perp} T P\| \ \forall T \in$

$B(H)$. The work was extended to a Von-Neumann algebra U and showed that there exists a constant $G \in B(H)$, dist(T, U) $\leq G \| \Delta_T \, |U^{'} \|$ where δ_T is the derivation $\delta_T(S) = ST - TS$ thus proving that the inequality holds for large classes of Von-Neumann algebras. Fong [25] considered $\lambda(M)$ defined as the smallest number $\|Z\|^2$ of Z that satisfy $[Z^*, Z] = M$ and showed that $1 \leq \lambda(M) \leq 2$ and $\lambda(M)$, M was suitably chosen if it is close to 2.

Matej [47] estimated the distance of $d_1 d_2$ to the generalized derivations and the normed algebra of $M^{'}$ and considered the cases when $M^{'}$ is an ultraprime, when $d_1 = d_2$ and $M^{'}$ are ultrasemiprime and when a Von Neumann algebra is $M^{'}$ from equation $\|M^{'}+N^{'}\| = \|M^{'}\|+\|N^{'}\|$, $M^{'}$, $N^{'} \in B(H)$. Abramovich, Aliprantis, Burkinshaw [3] showed that the point spectrum of S lies on the norm $\|S\|$; I is an identity operator on H only if the equation $\|I+S\| = 1+\|S\|$ is satisfied by operator S. Further, S, $U \in B(H)$ satisfy $\|S + U\| = \|S\| + \|U\|$ and zero which is the approximate point spectrum of the operator $\|S\|U - U\|S$ proved that for an isometric operator the converse is true for either S or U and a norm in $B(H)$ don't depend on the ideal on a norm of a derivation. Baxter [6] provided the supremum on $\|B^{-1}\|_2$ when the points $(y_i)^n{}_1$ form a subset of the integer L^d, and a conditional definite negative function ϕ of order 1, which included the multi-quadric for functions set that are large. Further, a constructive proof was provided that a minimum bound is not valid and a relevant method to analyze the problem on estimation of eigenvalues such an interpolation matrix was commented on.

The norm property coefficients was done by Cabrera, Rodriguez [15] for basic elementary operators $\|X_{c,d}\| \leq 2\|a\|\|b\|$, for Jordan elementary operator $U = \|X_{c,d}\| + \|X_{c,d}\|$ and $\|X_{c,d}\| + \|X_{c,d}\| \leq 2\|c\|\|d\|$ for the upper estimates. In fact, Gil [30] gave an estimate on matrix-valued function that is regular and showed that for normal matrices it is attainable then investigated their stability. Kittaneh [40] established the orthogonality, kernel and the range of a normal derivation with its association to opera-tors of norm ideals. Results relating to orthogonality of some derivation that are not normal were obtained. Stacho and Zalar [89] established the lower estimates for elementary operators of Jordan type in standard Banach algebras.

Danko [20] established that for all unitarily invariant norms and for bounded Hilbert space operators there holds $||C-D|^q| \leq 2^{q-1} |C|C|^{q-1} - D|D|^{q-1}|$, $q \geq 2$, if in addition, C and D are self-adjoint then $||||CX + XD|^q||| \leq 2^{q-1}||X||^{q-1}||||A|^{q-1}CX + XD|D|^{q-1}|||$, for all real $q \geq 3$.
C has a approximate point spectrum $\sigma_{ap}(A)$, has complex numbers

ω hence there exist $\{x_n\}_n \subseteq H$ which is a unit sequence such that $\lim_n \|C - \omega\|_{xn} = 0$. Since $\sigma_{ap}(C)$ is contained in the the boundary of $\sigma(C)$. Gustafson, Rao [31], $\|A\| \in \sigma(A)$ if and only if $\|A\| \in \sigma_{ap}(A)$ also $\sigma(A) \subseteq \overline{W(A)}$ (spectral inclusion) and if $\omega(A) = \|A\|$, then $\gamma(A) = \|A\|$. Therefore, the result implied that $\|A\| \subseteq \overline{W(A)}$ if and only if $\|A\| \in \sigma(A)$.

In fact, Megginson [51] established that $X \in \mathrm{K}$, then $\delta_B(X) \in \mathrm{J}$ and $\|BX - XB\|_{\mathrm{K}} = \|(B - \lambda)X - X(B - \alpha)\|_{\mathrm{J}} \leq 2\|B - \alpha\|\|X\|_{\mathrm{K}}$ for $\alpha \in \mathrm{C}$. Therefore, $\|\delta_B(X)\|_{\mathrm{K}} \leq 2d(B)\|X\|_{\mathrm{K}}$, depicting $\|\delta_B|\mathrm{K}\| \leq 2d(B)$. Fur-ther, the notion of R-universal operators was introduced and that R-universal is an operator $A \in B(H)$ if $\|\delta_B|\mathrm{K}\| = 2d(B)$ for every norm ideal $\mathrm{K} \in B(H)$. Landsman [45] proved that for a standard algebra oper-ator on H $\|M_{a,b}\| + \|M_{a,b}\| \geq 2(\sqrt{2} - 1)\|a\|\|b\|$. Therefore, both the lower norm and upper norm bounds have been established for normally represented elementary operators. Alexander [4] had an estimate on stochastic assumptions on transferring functions which are stable linearly and time-invariant systems. The approach of nonparametric minimax was adopted to measure estimate accurately, an estimator of quality was measured over a transfer functions of family with the worst case error. The polyno-mial and exponential decaying impulse response sequences families were taken into consideration. The finite impulse response approximation for upper non-asymptotic bounds on accuracy of the estimator of the least squares was established. It was established that the speed with which a true response with impulse is tending to null attained an estimate accu-rately was determined essentially. Estimation accuracy on lower bounds were presented and an adoptive estimator was developed which provided information about true systems that is not exploitative.

Shinji [86] established that for a holomorphic functions f with

$Re\{gf'(g)\} > \alpha$ and $Re\{gf''(g)/f'(g)\} > \alpha - 1$, $(0 \leq \alpha < 1)$ re-spectively in $\{|g| < 1\}$, estimates of $\sup_{|g|<1}(1 - |g|^2)|f''(g)/f'(g)|$ were given and functions Gelfer-convex of exponential order α, β was also con-

sidered. Milos, Dragoljub [52] considered elementary operators $x \to \sum_{j=1}^{n} v_j x w_j$ that acts on a Banach algebra, v_j and w_j denotes sepa-rate generalized scalar elements of commuting families. The ascent es-timation and lower bound estimation of an operator was given. Ad-ditionally, Fuglede-Putnam theorem for elementary operator is a weak variant with v_j and w_j are strongly commuting families were given i.e $v_j = v_j' + iv_j''(w_j = w_j' + w_j'')$, for all v_j' and v_j'' (w_j and w_j'') commutes.

Further, result concerning L^1 estimate in Fourier transform of a class C_{cpt}^{∞}

function in R^{2n} was obtained.

Barraa and Boumazgour [7] characterized that the norm of bounded operators more than one in a Hilbert space is the same summation of the norms which showed that $\delta_{S,A,B}$ is convexoid with the convex hull of its spectrum if and only if A and B are convexoid. Richard [78] established the CB-norms of elementary operators and the lower bounds for norms on $B(H)$. The result was concerned with the operator $U_{A,B}X = AXB+BXA$ which showed that $\|U_{A,B}\| \geq \|A\|\|B\|$ which proved a conjecture of Math-ieu, other results and formula of $\|U_{A,B}\|_{CB}$ and $\|U_{A,B}\|$ were established. Richard [79] provided the estimation on the norm of elementary opera-tors that are completely bounded was a direct proof which was possible in $B(H)$ through a generalized theorem by Stampfli [87] and it was shown that an operator J of length l equals to m-norm and $m = l$.

Seddik [81] proved that lower estimate bound $\|T_{M,N}\| \geq 2(\sqrt{2}-1)\|M\|\|N\|$ holds, if it satisfies one of the conditions: (i). A standard operator alge-bra on $B(H)$ is L and $M, N \in L$, (ii). L is ideally normed on $B(H)$ and $M, N \in B(H)$. Florin, Alexandra [26] estimated the norm of operator $H_{\theta,\lambda} = U_\theta + U_\theta^* + (\lambda/2)(V_\theta + V_\theta^*)$ which is an element on a C^*-algebra $A_\theta = C^*(U_\theta, V_\theta$ unitaries $: U_\theta V_\theta = e^{2\pi i\theta} V_\theta U_\theta)$. Further, proved for every $\lambda \in C$ and $\theta \in [\frac{1}{4}, \frac{1}{2}]$ the inequality

$$\|H_{\theta,\lambda}\| \leq \sqrt{4+\lambda^2-\left(1-\frac{1}{\tan \theta,\lambda}\right)\left(1-\sqrt{\frac{1+\cos^2 4\pi\theta}{2}}\right)min\{4, \lambda^2\}}.$$

This improved the significance of the inequality $\|H_{\theta,2}\| \leq 2\sqrt{2}, \theta \in [\frac{1}{4}, \frac{1}{2}]$, conjec-tured by Beguin, Valette and Zuk. Siva, Richard, Edwin [84] introduced a method to proof the estimate $\left\|\frac{d^2u}{dx_i dx_j}\right\|_{C^\alpha} \leq e\|t\|_{C^\alpha}$, and x solved the equation $\delta x-\beta x = t$. The technic is applicable to Laplacian on R^∞ and be used to obtain similar estimate when the Laplacian is replaced by elliptic operators or infinite-dimensional operators.

Gil [29] considered commuting matrices of matrix valued analytic func-tion and established a norm estimate, in particular, two matrices of matrix valued functions on a tensor product in a Euclidean space were explored. Stephen [90] communicated results on complex symmetric operator theory and showed that two non-trivial examples were of great use in studying Schr¨*o*dinger operators. To compute the norm of a compact complex symmetrical operator, a formula was proposed and the observation was applicable to problems which are related to quantum mechanics. Estimate was given on the density matrix of a single-particle for Schr¨*o*dinger oper-ators with spectral gaps and the exponential decay of the resolvent. New methods were provided to evaluate the resolvent norm for Schr¨*o*dinger operators appearing on complex scale theory in resonance.

Man-Duen, Chi-kwong [50] showed that triangle inequality served an up-per norm bound of an ultimate estimate for the sum operators that is sup$\{ T^{*}RT + V^{*}SV \| : T$ and $V \}$

are unitaries = min $\|R + \lambda I\| + \|S - \lambda I\| : \lambda \in$ C. The result discussed had relationship to normal dilations, spectral sets and the Von Neu-mann inequality. Yong, Toshiyuki [95] gave a norm estimate on pre-schwarzian derivatives of a specific type of convex functions by introduc-ing a maximal operator of independent interest of a given kind. The relationship between the convex functions and the Hardy spaces was discussed. Ola, Akataka, Derek [73] analyzed the structure of the set $D = \{y \in D(\delta) : \lim_{n\to\infty} \Delta_n(y) = \Delta(y)\}$ for convergence of the generators that are pointwise where α is an approximate inner flow on a C^*-algebra T with generator Δ and Δ_n be bounded generators of the approximate flows α^n. In fact, the relationship of D and various cores related to spectral subspaces were examined.

Seddik [82] showed that E is a normal operator which is invertible in $B(H)$ if the estimate $\|E \otimes E^{-1} + E^{-1} \otimes E\|_\lambda \leq \|E\|\|E^{-1}\| + \frac{1}{\|E\|\|E^{-1}\|}$ holds, such that $\|.\|_\lambda$ is a one-to-one norm on the tensor-product $B(H) \otimes B(H)$, when E is invertible self-adjoint then the equation becomes an equality. Further, the characteristics of $E \in B(H)$ satisfied the relation $\|E \otimes E^{-1} + E^{-1} \otimes E\|_\lambda = \|E\|\|E^{-1}\| + \frac{1}{\|E\|\|E^{-1}\|}$ then gave characterizations by inequalities or equalities of normal-operators in $B(H)$. Bonyo and Agure [10] characterized the norm ideal on norm of inner derivation to be equal to the quotient algebra and investigated them when the nor-mal and hyponormal operators are implementing them on norm ideals. Bonyo and Agure [11] investigated the relation between the inner deriva-tion implemented by Z on norm J and the numerical-range of an operator $Z \in B(H)$ with its diameter and considered application of T-universality on the relation.

Okelo, Okongo and Nyakiti [58] investigated the project tensor-product, $V_\Gamma' \otimes_\rho W_\Gamma'$ of these algebras. It was established that $\|\Delta_{S'}\| \leq \|\Delta^{(1)}_{S'} + \Delta^{(2)}_{S'}\| \leq 2\|\Delta_{S'}\|$ holds if $\lambda = \sum_i v_i' \otimes w_i'$ belongs to $A_\Gamma \otimes_\rho B_\Gamma$ and $\Delta_{S'}$ on λ is a norm attainable α-derivation given by $\Delta_{S'} = \Delta^{(1)}_{S'} + \Delta^{(2)}_{S'}$. Bonyo and Agure [9] gave the definition of inner-derivations implemented by A, B on $B(H)$ as $\delta_A(Y) = AY - YA$ $\delta_B(Y) = BY - YB$ and generalized derivation by $\delta_{A,B}(Y) = AY - YB$ $\forall Y \in B(H)$. Further, a relation between the norms of δ_A, δ_B and $\delta_{A,B}$ on $B(H)$ was specifically established when the operators A, B are S-universal.

Ber, Sukochev [8] showed that for every self-adjoint element $b \in S(N)$ a scalar $\lambda_0 \in$ R exists such that $\forall \varepsilon > 0$, then there exist a unital element u_ε from N satisfy $|[b, u_\varepsilon]| \geq (1 - \varepsilon)|b - \lambda_0 1|$. From this result a corollary is that for any derivation δ on N with the range on an ideal

I $\subseteq N$ the derivation δ is inner that is $\delta(.) = \delta_e(.) = [e, .]$ and $e \in$
I. Similarly the inner derivations on $S(M)$ results were also obtained. Pablo, Jussi, Mikael [75] provided theoretic estimate of two functions for the essential-norm as a composition-operator C_φ that acts on the space $BM\ OA$ (bounded mean oscillation for analytic functions); one in terms of the n-th power φ^n denoted by φ and the other involved the Nevanlinna counting function. Triet, Jianfeng [91] introduced a new type of norm for semimartangles, the defined norm of quasimartangales and then characterized the square integrable semimartangales. Therefore, the zero-sum stochastic differential games study was done and the value of the process was conjectured as semimartangale with probable class measures under some conditions.

Kingangi, Agure and Nyamwala [43] attempted the result on lower bound of the norms for finite dimensional operators. Odero, Agure, Rao [57] de-termined the norm of symmetric operator in an algebra which is two-sided. More precisely, investigated the injection of tensor norm through the lower bound of the operator. In addition, the irreducible C^*-algebra on the inner derivation norm was determined and Stampfli [87] confirmed the result for these algebras. Kinyanjui [41] estimated the norm-attainability for elementary-operators on inner derivation, generalized derivation, ba-sic elementary operator and Jordan-elementary operator under norms.

Wafula, Okelo and Ongati [93] studied normally represented operator which is a special type of elementary operator and results showed that elementary equals its largest single value that is $U_i(M) = \|M\|$ since $S_{A,B} = A \otimes hB + B \otimes hA$ is represented normally, then $\|S_{A,B}\|_{Inj} \geq \sqrt{-}\|A\|\|B\|$.

Elena, Lorenza, Ivan [24] studied properties of continuity of module spaces for operators of $\imath$-pseudo-differential $Op_\imath(c)$ in a Wiener amalgan space with a symbols c and obtained a bounded result for $\imath \in (0, 1)$ where $\imath = 0$ and $\imath = 1$ at end points and other operators were unbounded. In addition, it was exhibited the operator norm for the function $\imath \in (0, 1)$ has an upper bound which is independent on parameter $\imath \in (0, 1)$ was found. Jian-Feng, David [34] obtained R^1, R^2 and R^∞ norm of the operator K^*_0 and $R^p(D) \rightarrow R^\infty(D)$ norm of the operator C and J_0 provided $p > 2$. Since approximations of fixed in different space and classes have been done therefore, Okelo [70] discussed the approximate non-expansive operators on fixed points in Hilbert spaces. Particulary, it was proved that in an invariant subspace H_0 on a complex-Hilbert space H has a non-expansive retraction that is unique R of H_0 onto $\Gamma(Q)$ and $y \in H_0$ exists and a sequence $\{\xi_n\}$ generated by $\{\xi_n = \epsilon_n f(\xi_n) + (1 - \epsilon_n)T_{\xi_n} \xi_n$ for all $n \in N$ is strongly convergent to $T y$.

Cristina, Camil [19] proved the multilinear operators in R^d under vector-valued and mixed-norm estimates in multiple, precisely, the multilinear variables of the Hardy-Little wood, maximal function and the operators T_k associated with a single space along dimension k. It was shown that the input functions are not necessary in $L^p(\mathrm{R}^d)$ when the dimension $d \geq 2$ but can be elements of mixed-norm spaces $L^{p_1}_{x_1} \ldots L^{p_d}_{x_d}$. The purpose for this study is to establish norm-attainability conditions for derivations, to determine upper and lower norm-estimates for norm attainable derivations in Banach-algebras.

1.2 Basic concepts

In this section, the definitions that are basic and results on Hilbert space, field, vector space, norm, Banach space, numerical radius, numerical range, inner product, commutator and derivations are reviewed.

Definition 1.1 (93, Def. 1.1). A field K is a binary set operations that are additive and multiplicative that satisfy the axioms below:

(i). Are closed under additive and multiplicative: $w^{'}+v^{'} \in$ K and $w^{'}.v^{'} \in$ K $\forall w^{'}, v^{'} \in$ K.

(ii). Law of association : $w^{'} + (x^{'} + y^{'}) = (w^{'} + x^{'}) + y^{'} \quad \forall w^{'}, x^{'}, y^{'} \in$ K.

(iii). Commutativity: $w^{'}+v^{'} = v^{'}+w^{'}$ and $(w^{'}.v^{'}).y^{'} = (v^{'}.y^{'}).w^{'} \forall w^{'}, v^{'}, y^{'} \in$ K.

(iv). Additive and multiplicative identities: $\forall w^{'} \in$ K $\exists - w^{'} \in$ K : $w^{'} + -w^{'} = 0$. And $w^{'-1} \in$ K : $w^{'}.v^{'-1} = 1$.

(v). Distribution: $w^{'}(v^{'} + y^{'}) = (w^{'}v^{'} + w^{'}y^{'}) \ \forall w^{'}, v^{'}, y^{'} \in$ K.

(vi). Additive inverse: $\forall w^{'} \in$ K $\exists z^{'} \in$ K : $w^{'} + z^{'} = 0$ and $z^{'} + w^{'} = 0$ then $w^{'} = -z^{'} \ \forall z^{'}, w^{'} \in$ K.

(vii). Multiplicative inverse: For each $d^{'} \in$ K the equation $t^{'}.d^{'} = 1$ and $d^{'}.t^{'} = 1$, $t \in$ K is the multiplicative inverse written as $d^{'-1}$.

Definition 1.2 (76, Def. 1.2). Let $G^{'}$ be a vector space over a field F is a set which is non-empty with vector additive and multiplicative operations that are:

(i). Commutativity $c' + f' = f' + c',\ \forall c', f' \in G'$.

(ii). Associativity $c' + (f' + e') = (c' + f') + e'\ \forall c', f', e' \in G'$.

(iii). Additive inverse $\forall c' \in G',\ \exists - c' \in G' : c' + -c' = 0$.

(iv). Additive identity $\forall c' \in G',\ \exists 0 \in G' : c' + 0 = c'\ \forall c' \in G'$.

(v). Multiplicative identity $1.c' = c'\ \forall c' \in G'$.

(vi). Distributive property $\forall p' \in$ F and $\forall c', f' \in G', p'(c' + f') = (p'c' + p'f')$.

(vii). Law of Unitary $\forall c' \in G', 1.c' = c'$.

Definition 1.3 (63, Def. 2.1). A norm X' is a non-negative function $\|.\| : X' \rightarrow$ $R^+ \cup (O)$ sufficing the axioms below:

(i). $\|c'\| \geq 0\ \forall c' \in X'$.

(ii). $\|c'\| = 0$ only if $c' = 0\ \forall c' \in X'$.

(iii). $\|\alpha c'\| = |\alpha|\ c'\|;\ \forall c' \in X'$ and $\alpha \in$ C .

(iv). $\|c' + v'\| \leq \|c'\| + \|v'\|;\ \forall c', v' \in X'$.

The ordered pair $(X', \|.\|)$ is normed space.

Definition 1.4 (41, Def. 1.4). Banach space is a complete normed space.

Definition 1.5 (44, Def. 3.1-1). A map ., . : E x E → K is an inner product such that $\forall s', t', u' \in$ E, β, $\alpha \in$ K if it satisfy:

(i) $s', s' \geq 0$ and $s', s' = 0$, only if $s' = 0$.

(ii). $\beta s' + \alpha t', u' = \beta s', u' + \alpha t', u'$.

(iii). $s', t' = \overline{t', s'}$.

A pair (E, ., .) is a space called inner product.

Definition 1.6 (74, Def. 2.2). A Hilbert-space is a space with complete inner product.

Definition 1.7 (61, Definition 2.1). Let $T \in B(H)$, then,

(i). Numerical-range by $W'(T) = \{ Te, e : e \in H, \|e\| = 1\}$.

(ii). Numerical-radius by $\omega'(T) = \sup\{|s| : s \in W'(T)\}$.

Definition 1.8 (64, Def. 1.6). The spectrum P given by $\sigma(P) = \{P - \lambda I : \lambda \in C\}$ is not invertible.

Definition 1.9 (42, Def.1.13). An operator that is commuting with the adjoint is normal.

Example 1.10 (42). Example 1.14] Let $A : Y \longrightarrow Y$ and $A = 2iI$, I is an identity and A is normal then $AA^* = A^*A = I$.

$$
\begin{aligned}
AA^* &= (2iI)(2iI)^* \\
&= (2iI)(-2iI) \\
&= -4i^2 I \\
&= 4I
\end{aligned}
$$

$$
\begin{aligned}
A^*A &= (2iI)^*(2iI) \\
&= (-2iI)(2iI) \\
&= -4i^2I \\
&= 4I
\end{aligned}
$$

It follows that $AA^* = A^*A$.

Definition 1.11 (76, Def. 1.1). Linear operators are mappings $T^{'} : X^{'} \rightarrow Y^{'}$ for:

(i). $M^{'}(c^{'} + f^{'}) = M^{'}(c^{'}) + M^{'}(f^{'})\ \forall c^{'}, f^{'} \in Y^{'}$.

(ii). $M^{'}(\alpha f^{'}) = \alpha M^{'}(f^{'})\ \forall f^{'} \in Y^{'}$ and complex numbers α.

(iii). $K > 0$ is constant such that $\| M^{'} f^{'} \leq K \| f^{'} \|\ \forall f^{'} \in Y^{'}$ then $M^{'}$ is bounded.

Definition 1.12 (66, Def. 2.4). An operator is self-adjoint if $S = S^*$.

Definition 1.13 (67, Def. 3.1). For an operator K there exist a unit vector $t \in H$ such that $\|Kt\| = \|K\|$ is norm-attainable.

Definition 1.14 (59, Def. 1.1). A Banach algebra is a normed algebra if it is complete.

Definition 1.15 (1, Def. 1.5). Banach $*$-algebra $R^{'}$ is a C^*-algebra if $\|r^{'} r^{'*}\| = \|r^{'}\|^2\ \forall r^{'} \in R^{'}$.

Definition 1.16 (65, Definition 2.1). Elementary operator $T : B'(H) \rightarrow B'(H)$ is defined by $T_{D_i,E_i}(Y) = \sum_{i=1}^{n} D_i Y E_i \ \forall Y \in B'(H)$ and $\forall D_i, E_i$ fixed in $B'(H)$ where $i = 1, ..., n$.

(i). Left-multiplication operator $L_D : B'(H) \rightarrow B'(H)$, $L_D(Y) = DY, \ \forall Y \in B'(H)$.

(ii). Right-multiplication operator $R_E : B'(H) \rightarrow B'(H)$, $R_E(Y) = YE, \ \forall Y \in B'(H)$.

(iii). Generalized-derivation, $\delta_{D,E} = L_D - R_E$.

(iv). Inner derivation (implemented by D), $\delta_D(Y) = DY - YD$.

(v). Basic elementary operator (implemented by D, E), $M_{D,E}(Y) = DYE, \ \forall Y \in B'(H)$.

(vi). Jordan-elementary operator, $U_{D,E}(Y) = DYE + EYD, \ \forall Y \in B'(H)$.

Definition 1.17 (62, Def. 1.11). An operator Q a projection if $Q^2 = Q$.

Definition 1.18 (87, Def.). A derivation is a mapping $P' : U' \rightarrow U'$ which satisfy $P'(c'd') = c'P'(d') + P'(c')d'$ for all $c', d' \in U'$.

Definition 1.19 (69, Def. 1.2). An operator S is a maximal numerical range defined as $W_0(S) = \{\beta : St, t \rightarrow \beta$, where $\|t\| = 1$ and $\|St\| \rightarrow \|S\|\}$.

1.3 Statement of the problem

Since H is a Hilbert space whose dimension are infinite with the algebra of linear operators on H being $B^{'}(H)$, then algebras of norm-attainable operator on H is $NA(H)$. Norm-attainable conditions for elementary operator has been done and results obtained. But, norm-attainable con-ditions for derivatives in Banach algebras and norm-estimates which is upper and lower norm estimates for derivations in Banach algebras has not been investigated. Objectively, the study will: establish norm attain-ability conditions for derivations in Banach algebras and determine the upper and the lower norm estimates for norm-estimates for norm attain-able derivations in Banach algebras. In this study therefore, we seek to determine the norms of derivations as an example of elementary operator when implemented by norm-attainable operator.

1.4 Objectives of the study

Are to:

(i). Establish norm attainability conditions for derivations in Banach algebras.

(ii). Determine the upper and the lower norm-estimates for norm attain-able derivations in Banach algebras.

1.5 Significance of the study

Norm-attainability has been investigated by many mathematicians for a long time from the related literature. The result obtained from this study will be helpful in comprehending the patterns of electrons movement in orbits and approximating the distances moved in quantum mechanics. The result will also be useful in solving integral equations to obtain results for bounded domain for instance $f'(x') = \int_a^b x'(t')dt'$ if f' is bounded and has norm $\|f'\| = b - a$, $t' \in J = [a, b]$ then $|f'(x')| = |\int_a^b x'(t')dt'| \leq (b - a)max_{t' \in J} |x'(t')| = (b - a)\|x'\|$.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, we review related literature on norm-attainability and norm-estimates for norm attainable derivations in Banach algebras.

2.2 Norm-attainability

Stampfli [87] determined the inner derivation $\delta_{T0} : A_0 \to T_0A_0 - A_0T_0$ which acts on Banach algebra $B(H)$ on Hilbert space H. Further, $\| \delta_{T0} \| = \inf 2 \| T_0 - \lambda I_0 \|$ for every complex λ was shown. For a normal T, then $\|\delta_{T0} \|$ can be expressed as the geometry of the spectrum of T_0.

Lemma 2.1. *[87 Lem. 1] If* $\|T_0\| = \|x\| = 1$ *and* $\|T_0x\|^2 = (1 - \varepsilon)$, *then* $\|(T_0{}^*T_0 - 1)x\|^2 5\ 2\varepsilon$.

In lemma 2.1 the lower and upper norm estimate on a norm of a derivation are determined. In this study we have determined lower and upper norm-estimates for norm attainable derivations in Banach algebras.

Theorem 2.2. *[87 Thm. 1]* $\|\delta_{T_0}\| = 2\|T_0\|$ *if and only if* $0 \in W_0(T_0)$.

Theorem 2.2 establishes the norm of inner derivation on a maximal numerical range of operator T_0. In this study we have determined the norm estimate for norm-attainable derivations in Banach algebras.

Johnson [37] established method which apply to a uniform convex spaces with a large class, that is the formula $\|\delta_T\|$ is false in l^p and $L^p(0, 1)$ $1 < p < \infty$, $p \neq 2$.

Proposition 2.3. *[37, Prop. 2] Let U be a normed space on K then* $u, v \in U$ *have the following properties.*

(i). $\|u\| = 1$ *and there exists* $g \in U^*$ *with* $\|g\| = 1$ *such that* $\{u_n\}$ *is a sequence with* $\|u_n\| \leq 1$, $g(u_n) \to 1$ *then* $u_n \to u$.

(ii). $\|t\| = 1$ *and the unit ball* U_1 *is uniformly convex at t.*

(iii). For every $\lambda \in K$, $\|u + \lambda v\| < 1$.

(iv). $\forall \lambda \in K$, $\|v + \lambda u\| = 1$.

Proposition 2.3 determines the norm estimate in a uniform convex space. In this study we have established the norm estimate for norm-attainable derivations in Banach algebras.

Proposition 2.4. *[37, Prop. 3] Let* F' *be a uniform convex Banach-space* f', $s \in F'$ $j, k \in F'^*$ *with* $\|f'\| = \|s\| = \|g\| = \|k\| = g(f') =$

$k(f') = 1$, $k(f') = 0$, $g(s) = 0$ *and suppose g is the only element h of* F'^* *with* $\|h\| = h(f') = 1$. *Then conditions in proposition 2 are satisfied by* f', s, g.

Proposition 2.4 considers the norm on a convex Banach space but we have obtained norm-estimates for norm attainable derivatives in Banach-algebras.

Johnson [36] found that a derivation in $B(H)$ is a map $\delta : B(H) \rightarrow B(H)$ with $\delta(P\,S) = P\,\delta(S) + \delta(P\,)S\,P$, $S \in B(H)$. Such derivations are necessarily continuous and if $S \in B(H)$ then $\delta_S(P\,) = P\,S - SP$ is a derivation in $B(H)$.

Theorem 2.5. *[36, Thm. 1] If a derivation* Δ *is in* $B(H)$ *then* $\Delta = \Delta_S$ *for some* $S \in B(H)$.

Theorem 2.5 found the norm of a derivation but we have established the lower and upper norm-estimates for norm attainable derivations in Banach algebras.

Theorem 2.6. *[36, Theorem 2] (Stampfli [87])* $\|\delta_S\| = 2dist(S, \mathbb{C}I)$.

Theorem 2.6 determines the inner derivation which is equivalent to $2dist(S, \mathbb{C}I$ but we have established the lower and upper norm-estimates for norm at-tainable derivations in Banach algebras.

In Gajendragadka [27] was concerned with computation of norm of deriva-tion and the Von-Neumann algebra. Specifically when the Von-Neumann algebra act on separable Hilbert space H, $T \in \mathrm{U}$ was proved that then δ_T is the derivative induced by T, then $\|\delta_T\,|\mathrm{U}\| = 2\inf \|T - Z\|$, Z in the centre U.

Lemma 2.7. *[27 Lem. 1] If* U *is Von-Neumann algebra which act on Hilbert space H and if T is in* U*, then there exist* Z_0 *in* Z(U) *such that, for every projection P in* Z(U), $\|(T - Z_0)P\| = \inf \|(T - Z)P\|, Z \in Z(U)$.

Lemma 2.7 determines the on norm of a Von-Neumann algebra but we established lower and upper norm-estimates for norm attainable derivations in Banach algebras.

Theorem 2.8. *[27, Thm. 1] Let* U *be a Von-Neumann algebra on H and assume that* U' *is abelian. Then for T in* U*, there exist* Z_0 *in* $Z(U) = U'$ *such that* $\|\delta_T |U\| = 2\|T - Z_0\|$.

Theorem 2.8 found the norm of inner derivation but we have determined the norm of generalized derivation for norm-attainable derivations in Ba-nach algebras.

Kyle [38] examined the numerical-range of inner derivation and the ele-ment implementing it and the relationship between them.

Lemma 2.9. *[38, Lem. 2.1] For any Banach-algebra C,* $D(B, C) = D(L_B; MC) = D(R_B; M(C))$ *where* $L_B(X) = BX$ *and* $R_B(X) = BX$.

Lemma 2.9 established the numerical range in a complex unital Banach algebra but we have considered derivations in norm-attainable derivations.

Theorem 2.10. *[38, Thm. 2.3] Let* $A = LX$ *, for some Banach-space* X *, let* $\delta_{A,B}(X) = AX + XB$*. Then* $Q(\delta_{A,B}; L(LX)) = Q(A; LX) + Q(B; LX)$.

Lemma 2.10 established the generalized derivation but we considered in-ner derivation in norm-attainable operators.

Kyle [39] studied on norms of inner derivations and used their properties and concluded that a C^* algebra is a closed subset of all derivations which forms the inner derivations set and obtained the result which was a converse by Stampfli [87].

Proposition 2.11. *[39, Prop. 2.2] If x, l, y, k and* $A(l) = k(l)x$, *then* $\|A\| = inf\{ A + \lambda I\| : \lambda \in \mathbb{C}\} = 1$.

Proposition 2.11 investigates the norm of elementary operator but we considered norm attainability conditions for derivations in Banach-algebras.

Corollary 2.12. *[39, Cor. 2.1] Let* $\|\delta_A\| = 2inf\{ A + \lambda I\| : \lambda \in \mathbb{C}\}$ *for all A in* $L(X)$ *then* $\|\delta_A\| = 2$.

In corollary 2.12 the norm of inner derivation was found but we have investigated the norm of generalized derivation for norm-attainable derivations in Banach algebras.

Charles and Steve [16] answered the question when $X = T$ by structure characterization of compact derivations of C^*-algebras. Moreover, the structure of weak compact derivations of C^*-algebras was determined and as immediate corollaries of these results, conditions that were necessary and sufficient were obtained so that C^*-algebras admits non-zero compact or weakly compact derivation.

Lemma 2.13. *[16, Lem. 2.1] Let an infinite dimensional Hilbert-space be H, the algebra of all bounded linear operators on H be* $B(H)$. *If* δ *is a compact derivations of* $B(H)$ *then* $\delta \equiv 0$.

Lemma 2.13 gives the condition for $B(H)$ to be a compact derivation,

$\delta \equiv 0$ but we have given norm attainability conditions for norm-attainable derivatives in Banach-algebras.

Erik [23] established that any C^* algebra F on a Hilbert-space H with cyclic vector with property that to derivation δ of F into $B(H)$ an oper-ator y existed in $B(H)$: $\forall f \in F,\ \delta(f) = [y, f] = yf - fy$.

Theorem 2.14. *[23, Thm. 4.1] Let B be a C^* algebra, T a C^* subal-gebra and a derivation δ of T into B. For any finite set $t_1, ..., t_n$ in T,*
$\|\sum_{i=1}^{n} \delta(t_i)^*\delta(t_i)\| \leq 14\|\delta\|^2\|\sum_{i=1}^{n} t_i^* t_i\|$.

Theorem 2.14 determined the upper estimate for inner derivations but we have given norm attainability conditions for derivatives in Banach-algebras.

Corollary 2.15. *[23, Cor. 5.4] Let a C^*-algebra on a Hilbert-space H be T. Suppose T has a cyclic vector, then:*

a) For every operator y in $B(H)$, $d(y, T') \leq 12\|ad(y)|T\|$.

b) Any derivation δ of T into $B(H)$ implemented by an operator y is such that $\|y\| \leq 12\|\delta\|$.

Corollary 2.15 established upper estimate for C^* algebra derivations but we considered inner and generalized derivations.

Mathieu [48] proved that for bounded derivations that are non-zero then the product of two prime C^*-algebras are bounded.

Theorem 2.16. *[48, Thm. 1] Let δ be a derivation which is densely defined on a C^* algebra B. If δ^2 is bounded, then δ is also bounded.*

Theorem 2.16 investigated derivation of a C^*-algebra but we have considered inner and generalized derivations in Banach algebras.

Lemma 2.17. *[48, Lem. 1] Let Y be a prime C^* algebra. For all c, d, e $\in Y$, let $M_{c,d,e}$: YXY→ Y is the bilinear mapping* (z, w) *7→czdwe. Then* $\|M_{c,d,e}\| = \|c\|\|d\|\|e\|$.

Lemma 2.17 discussed the basic elementary operator and its norm but we have established norms of inner and generalized derivations in Banach algebras.

Volker [92] two automatic continuity problems for derivations on commutating Banach algebras were discussed : (a) Derivation on a commutative algebra is mapped onto the radical, and: (b) Banach algebras are contin-uous on semiprime derivations. It was proved that (b) implies (a). Fur-thermore, (b) proved that for special cases Banach algebras are reduced to a small class and also similar results were given on epimorphisms. In fact, it was shown that semisimple Banach algebras were characterized with no topologically nilpotent element other than zero being among the commutative Banach algebras; known examples of discontinuous deriva-tions on commutating Banach algebras depended majorly on the existing nontrivial nilpotent elements which was on a generalized derivation of semiprime Banach algebra and that nilpotent elements are continuous on a commutative Banach algebra without nontrivial.

Theorem 2.18. *[92, Thm. 6] The following four statements are equiva-lent.*

(i). The derivation on a nilpotent separating space has a commutative Banach algebra.

(ii). The derivation is continuous on a semiprime Banach algebra.

(iii). The derivation which is an integral domain is continuous on a Ba-nach algebra.

(iv). The derivation is continuous on a topological simple, commutative Banach algebra other than C.

Theorem 2.18 gave equivalent statements on derivations on a Banach algebra but we have given conditions for norm-attainability for derivations on Banach algebras.

Douglas [21] continued the study of $W_s(Y)$ which was considerably more amenable where Archbold [1] defined the smallest numbers to be $[0, \infty]$ and introduced two constants $W(Y)$ and $W_x(Y)$ such that $d(y, Q(Y)) \leq W(Y)\|D(y, Y)\| \ \forall y \in Y$ and $d(y, Q(Y)) \leq W_s(Y)\|D(y, Y)\| \ \forall y = y^* \in Y$.

Lemma 2.19. *[21, Lem. 4.1] Let M be a C^*-algebra and let $m \in M$. Then* $\|D(m, M)\| = \sup\{ D(mp, T/M)\| : P \in prime(M)\}$.

Lemma 2.19 discussed the upper norm estimate for distance but we considered norm of derivations in Banach algebras.

Corollary 2.20. *[21, Cor. 4.3] Let X be a C^* algebra with an identity and $x \in X_{sa}$. Then* $\|D(x, X)\| = \sup\{ \mu(xp) - \beta(xp)\| : P \in prime(X)\}$.

Corollary 2.20 discussed the norms of elementary operators but we considered norm estimates for derivations in Banach algebras.

Dutta, Nath, Kalita [22] showed that if α_1 and α_2 are δ-derivation and δ'-derivation on (T, γ) and (T', γ') and an arbitrary element $n = \sum_1 y_1 \otimes x_1$

of $(T, \gamma) \otimes_\rho (T', \gamma')$, then a derivation D on $\delta \otimes \delta'$ exists in $(T, \gamma) \otimes_\rho (T', \gamma')$ satisfy $\alpha(n) = \sum 1[(\alpha y_1 \otimes x_1) + y_1 \otimes (\alpha x_1)]$ in which many enlightening properties were investigated on $\|\alpha\| = \|\alpha_1\| + \|\alpha_2\|$ and $sp(\alpha) = sp(\alpha_1) + sp(\alpha_1)$.

Theorem 2.21. *[22, Thm. 2.1] Let α_1 and α_2 be δ-derivation and δ'-derivation on (P, Γ) and (P', Γ') respectively. Then*

(i). There exists a bounded $\delta \otimes \delta'$-derivation α on $(P, \Gamma) \otimes_\rho (P', \Gamma')$ defined by $\alpha(n) = \sum 1[(\alpha y_1 \otimes x_1 + y_1 \otimes \alpha x_1)]$, for each vector $n = \sum_1 y_1 \otimes x_1 \varepsilon (P, \Gamma) \otimes_\rho (P', \Gamma')$.

(ii). If D_1 and α_2 are $\delta-$ and δ'-inner derivation implemented by the elements $r_0 \varepsilon P$ and $s_0 \varepsilon P'$ respectively then α is an $\delta \otimes \delta'$-inner derivation implemented by $r_0 \otimes 1_\alpha + 1_\alpha \otimes s_0$.

(iii). If α_1 and α_2 are $\delta -$ and δ'-Jordan derivations, then $\delta \otimes \delta'$-Jordan derivation.

(iv). If (P, Γ) and (P', Γ') are involutive Gamma-Banach algebras, and if α_1 and α_2 are $\delta-$ and δ'-star derivations, then $\alpha \otimes \alpha'$-star deriva-tion.

Theorem 2.21 established the conditions for inner, Jordan, star deriva-tions but we considered generalized derivations in Banach algebras.

Theorem 2.22. *[22, Thm. 2.2] The following results are true :*

(i). If α is a derivation on $(T, \Gamma) \otimes_{rho} (T', \Gamma')$ such that $\alpha(\sum_1 y_1 \otimes x_1) = \sum z_1 \otimes x_1$, $z_1, x_1 \varepsilon T$ and x_1 s are $\delta \otimes$-idempotent elements of P', then

there exist an δ'-derivation α_1 on T defined by $\alpha_1 y \otimes x = \alpha(y \otimes x)$ for all $y \varepsilon T$ for every δ'-idempotent element $x \varepsilon P'$;

(ii). If α is bounded then α_1;

(iii). If α is an $\delta \otimes \delta'$-inner derivation implemented by an element m of the form $m = \sum_1 y_1 \otimes x_1$, where x_1's are δ'-idempotent elements, then α_1 is an δ-inner derivation implemented by $\sum_1 y_1$;

(iv). If (T, Γ) and (T', Γ') are involutive Gamma-Banach algebras, α α_1 are star derivations.

(v). If α is an $\delta \otimes \delta'$-Jordan derivation then D_1 is a δ-Jordan derivation;

(vi). If α is an $\delta \otimes \delta'$-derivation on $(T, \Gamma) \otimes_\rho (T', \Gamma')$ such that $\alpha(\sum_1 y_1 \otimes x_1) = \sum_1 y_1 \otimes s_1$, for δ idempotent elements y_1 in T, and $s \varepsilon T'$ then there exists an δ'-derivation α_2 on (T', Γ') given by $y \otimes \alpha_2 = \alpha(y \otimes x)$ for every δ-idempotent element $y \varepsilon T$ and for all elements $x \varepsilon T'$. The above results (ii), (iii), (iv) and (v) are true for α_2.

Theorem 2.22 established the conditions for inner, derivations but we have considered generalized derivations in Banach algebras.

Rajendra, Kalyan [77] showed that for the *n*th order commutator

$[[[k(B), Y], Y], ..., Y]$ a formula was obtained in terms of the Frechet derivatives $S^m k(B)$ in which the formula illustrated was used to obtain bounds for norms of a generalized commutator $k(B)Y - Y\,k(B)$ and their higher order analogues.

Theorem 2.23. *[77, Thm. 2.1] Let k be a holomorphic function on a complex domain* Ω *and let B be an operator contained in spectrum of* Ω*. Then* $k(B)Y - Yk(B) = \delta B(A)(BY - YB)$ *holds for all Y.*

Theorem 2.23 discussed the use of the formula to obtain the derivative of the inner commutators but we have established the norms of inner derivations in Banach algebras.

Theorem 2.24. *[77, Thm. 2.2] Let k be a continuously differentiable function on an open interval I. Then* $k(S)Y - Yk(S) = \delta k(S)(SY - YS)$ *holds for all self-adjoint operators S with their spectra in I, and for all skew-Hermitian operators Y.*

Theorem 2.24 discussed the derivative of inner commutators of self-adjoint operators and skew-Hermitian operators. In this study we have discussed inner and generalized derivations in norm-attainable operators.

Hong-Ke, Yue-qing [33] proved that $\sup\{\|\sum_{i=1}^{n} R_i Y S_i\| : Y \in B(H), \|Y\| \leq 1\} = \sup\{\|\sum_{i=1}^{n} R_i T S_i\| : UU^* = T^*U = I, U \in B(H)\}$. Therefore, there exists an operator Y_k which proved that $\|Y_k\| = 1$ so that $\|\sum_{i=1}^{n} R_i Y_k S_i\| = \sup\{\|\sum_{i=1}^{n} R_i Y S_i\| : Y \in B(H), \|Y\| \leq 1\}$ if and only if there exists a unitary $U_0 \in B(H)$ so that $\|\sum_{i=1}^{n} R_i U_0 S_i\| = \sup\{\|\sum_{i=1}^{n} R_i Y S_i\| : Y \in B(H), \|Y\| \leq 1\}$.

Corollary 2.25. *[33, Cor. 2.2] If the elementary-operator* $\delta_{\tilde{P},\tilde{Q}}$ *is norm attainable, then there exists an isometry or co-isometry* V_0 *such that*

$$\|\delta_{\tilde{P},\tilde{Q}}\| = \|\sum_{i=1}^{n} P_i V_0 Q_i\|.$$

Corollary 2.25 gave norm-attainability conditions for generalized derivations but we have considered norm-attainability conditions for inner derivations.

Lemma 2.26. *[33, Lem. 2.5] For an operator $P \in B(H)$ then P is norm attainable if and only if its adjoint P^* is norm attainable.*

Lemma 2.26 discussed the conditions for norm-attainable operators but we have considered conditions of norm-attainable derivations in Banach algebras.

Okelo, Agure and Ambogo [61] established the norm of Jordan-elementary operator $U_{M,N} : B(H) \to B(H)$ given as $U_{M,N} = MYN + NYM$, $\forall Y \in B(H)$ and M, N fixed in $B(H)$ and showed that $\|U_{M,N}\| \geq \|M\|\|N\|$ and then characterized the norm-attainable operators using this norm.

Theorem 2.27. *[61, Thm. 3.4] An operator $C \in B(H)$, C is norm attainable if and only if its adjoint is norm attainable.*

Theorem 2.27 determines the conditions of norm-attainable elementary operators but we have established the conditions of norm-attainability for derivations in Banach algebras.

Theorem 2.28. *[61, Thm. 5.1] Let $T_{N,C,D} : B(H) \to B(H)$, $Y \to CYD + DYC$, $\forall Y \in B(H)$ norm-attainable Jordan elementary oper-ator. Assume $C, D \in B(H)$ are norm attainable such that $C = \gamma Q$ and $D = \gamma R$ where $Q = |C|$, $R = |D|$ and γ a unitary in $B(H)$ then $\|T_{N,C,D}|B(H)\| \geq \|C|\,D\|$.*

Theorem 2.28 considers lower estimate for norm-attainable Jordan elementary operators but we considered lower and upper norm-estimates for norm attainable derivations in Banach-algebras.

Okelo [68] investigated that ideals of norm-attainable elements implemented by inner derivations of a C^*-algebra has relation to primitive ideals. Since there is a relationship between the constants $A(\xi)$ and $A_s\xi$ ideals of C^*-algebras and ideals that are primitive then related results were given.

Lemma 2.29. *[68, Lem. 3.1] Let ξ be a C^*-algebras, prim (ξ) the set of all primitive ideals in ξ, $[A]$ the cannonical image of A in ξ/K, then δ_A^N , $\|\delta_A^N\|$ = $\sin\{\delta_{[A]}|(\xi)/K\| : K \in prim(\xi)\}$ is a norm-attainable inner derivation*

Lemma 2.29 determined the norm of norm-attainable inner derivations but we have given norm-attainability conditions for inner derivations.

Corollary 2.30. *[68, Cor. 3.2] Let $A \in \xi$ be norm attainable and con-sider the norm attainable inner derivation δ_A^N , induced by A. Then the following hold:*

(i). δ_A^N is self adjoint.

(ii). A is normal.

(iii). $\|\delta_A^N{}_{*}|\xi\| = 2d(A^*)$.

Corollary 2.30 investigated norm-attainability conditions for inner derivations but we have considered generalized derivations.

Okelo, Agure and Oleche [66] gave results on necessary and sufficient conditions for norm-attainable operators also studied norm-attainable oper-ators and generalized derivations

Theorem 2.31. *[66, Thm. 3.3] An operator $B \in B(H)$ implemented by norm-attainable inner derivation, ${\delta_B}^N$ is uniformly dense in $B(H)$.*

Theorem 2.31 establishes norm-attainability condition for inner deriva-tion which is uniformly dense in $B(H)$ but we have considered norm-attainability conditions generalized derivation in Banach algebras.

Lemma 2.32. *[66, Lem. 3.5] The set of operators B, $C \in B(H)$ which are implemented by norm-attainable generalized derivation, ${\delta_{B,C}}^N$ are uniformly dense in $B(H)XB(H)$.*

Lemma 2.32 establishes norm-attainability condition for generalized derivation which are uniformly dense in $B(H)$ X $B(H)$ but we have considered norm-attainability conditions of inner derivation in Banach algebras.

Okelo [65] extended the work by presenting new results on conditions that are sufficient and necessary for norm-attainable operators on Hilbert space, elementary operator and generalized derivation was established. Further, Okelo [65] established that a unit vector exists $\lambda \in H$, $\|\lambda\| = 1$ so that $\|S\lambda\| = \|S\|$ with $S\lambda, \lambda = \eta$.

Theorem 2.33. *[65, Thm. 2.1] Let $Z \in B(H)$, $\beta \in W_0(Z)$ and $\phi > 0$. There $\exists$ an operator $T \in B(H)$ such that $\|Z\| = \|T\|$, with $\|Z - T\| < \phi$. Furthermore, there exists a vector $\lambda \in H$, $\|\lambda\| = 1$ such that $\|T\beta\| = \|T\|$ with $T\lambda, \lambda = \beta$.*

Theorem 2.33 determines the norm of an operator in a maximal numerical range of Z but we established norm-attainable conditions for derivations in Banach algebras.

Lemma 2.34. *[65, Lem. 3.1] Let* $S \in B(H)$*.* δ_S *is norm attainable if there* $\exists$ *a vector* $\gamma \in H$ *such that* $\|\gamma\| = 1$, $\|S\gamma\| = \|S\|$, $S\gamma, \gamma = 0$.

In lemma 2.34 norm-attainability for inner derivations is established but we have considered norm-attainable operators for generalized derivations in Banach algebras.

Okelo [62] studied norm-attainable operators that are convergent and established projective tensor norm via norm-attainable operators.

Theorem 2.35. *[62, Thm. 3.1] Let* $M \in B(H)$*,* $\alpha \in W_0(M)$ *and* $\mu > 0$*. There exists an operator* $N \in B(H)$ *such that* $\|M\| = \|N\|$*, with* $\|M - N\| < \mu$*. Furthermore, there exists a vector* $\theta \in H$*,* $\|\theta\| = 1$ *such that* $\|N\theta\| = \|N\|$ *with* $Z\theta, \theta = \alpha$*.*

Theorem 2.35 investigates upper norm estimate of a maximal numerical range $\beta \in W_0(A)$ but we established lower and upper norm estimate for norm-attainable derivations in Banach algebras.

Theorem 2.36. *[62, Thm. 4.4] Let* $\{K_n\}$ *and* $\{L_n\}$ *be sequences of operators in* $NA(H)$ *and* $NA(H')$ *respectively. If one converges to zero uniformly and the other is bounded, then* $\{K_n \otimes L_n\}$ *converges to zero uniformly.*

Theorem 2.36 shows that sequence of $NA(H)$ converges uniformly to zero but we established norm-attainability conditions for derivations in Banach algebras.

Sayed, Madjid, Hamid [80] proved that for a linear map $\Delta : U \to U$, $\Delta(XY) = \Delta(X)Y + \Delta X(Y)$ for each $X, Y \in U$ is a derivation, then any two derivations Δ and Δ' on a C^*-algebra U exists a derivation $\delta \in$ U such that $\Delta\Delta' = \delta^2$ if and only if either $\Delta' = 0$ or $\Delta = f\Delta'$ for any $f \in$ C.

Proposition 2.37. *[80, Prop. 2.4] Let D be a subalgebra of M_2C which is a generation of E_{11} and E_{12} and δ, δ' be derivations on D. Then $\Delta\Delta' = \delta^2$ if and only if $\Delta' = 0$ or $\Delta' = \Delta_{\alpha' E12}$, $\alpha' \in$ C implies that $\Delta = \Delta_{\alpha E12}$, α $\in$ C, or equivalently $\Delta' = 0$ or $\Delta'^2 = 0$ implies $\Delta^2 = 0$.*

Theorem 2.37 discussed the product of two derivations in a subalgebra matrix. This study we have established norm attainability conditions for derivations in Banach-algebras.

Theorem 2.38. *[80, Thm. 3.1] Let* U *be a C^*-algebra and Δ, Δ' be derivations on* U*. Then there exists a derivation δ on* U *such that $\Delta\Delta' = \delta^2$ if and only if either $\Delta' = 0$ or $\Delta = f\Delta'$ for all $f \in$* C.

Theorem 2.38 discussed two derivations in a C^*-algebra and their product. This study we established norm-attainability conditions for derivations in Banach algebras.

Clifford [18] studied hypercyclic generalized derivations acting on separa-ble ideals of operators then identified concrete examples and established some conditions that are necessary and sufficient for their hypercyclic-ity. Particular Banach algebras acted on by the dynamics of elementary operators were considered.

Theorem 2.39. *[18, Thm. 4.1] Let X and Y be hyponormal such that X, Y $\in$ B(H). The generalized derivation $\tau_{X,Y} : C_2 \rightarrow C_2$ is not super-cyclic.*

Theorem 2.39 discussed supercyclicity of generalized derivation but we have investigated inner derivations on Banach algebras.

Oyake, Okelo and Ongati [74] characterized inner derivations in Banach algebra and investigated inner derivation properties that are implemented by norm-attainable operators such as measurability, normality continu-ity, linearity, trace and spectra of inducing operator and determined the norms. The result showed that the derivations admitted tensor norms of operators.

Theorem 2.40. *[74, Thm. 3.2] Let* $V : H_1 \rightarrow H_2$ *and* $W : K_1 \rightarrow K_2$ *be bounded operators between Hilbert-spaces. Then a unique bounded operator* $V \hat{\otimes} W : H_1 \otimes K_1 \rightarrow H_2 \otimes K_2$ *exists such that* $(V \otimes W)(X \otimes Y) = VX \otimes WY$ $\forall X \in H$ *and* $\forall Y \in K$*. Moreover,* $\|V \hat{\otimes} W\| = \|V\|\|W\|$

Theorem 2.40 investigates the norm of bounded operators between Hilbert spaces but we have given norm attainability conditions for derivatives in Banach-algebras.

Kinyanjui [42] characterized norm-attainable elementary operator and showed if operators *M, N* and $\delta_{M,N}$ are norm attainable then $\delta_{M,N}$ is normally represented.

Lemma 2.41. *[42, Lem. 2.7.] Let an infinite dimensional complex non-separable Hilbert space be H and the algebra of all bounded linear operators on H be* $B'(H)$*. Let* $\delta' : B'(H) \rightarrow B'(H)$ *defined as* $\delta_{P'}(Y') = P'Y' - Y'P'$*. Then* $\delta_{P'}$ *is norm attainable.*

Lemma 2.41 establishes conditions of norm-attainability for inner derivations but we have established conditions of norm-attainability for a generalized derivations in Banach algebras.

Lemma 2.42. *[42, Lem. 2.10] Let an infinite dimensional complex non-separable Hilbert space be H and the algebra of all bounded linear operators on H be* $B^{'}(H)$. *Let* $\delta : B^{'}(H) \rightarrow B^{'}(H)$ *be defined as* $\delta_{P^{'},Q^{'}}(X^{'}) = P^{'}X^{'} - X^{'}Q^{'}$. *Then* $\delta_{P^{'},Q^{'}}$ *is norm-attainable if* $P^{'}$ *and* $Q^{'}$ *are norm-attainable.*

Lemma 2.42 establishes conditions of norm-attainability for generalized derivations but we have established conditions of norm-attainability for inner derivations in Banach algebras.

In Okelo, Aminer [67] norm inequalities that are new of matrices of norm-attainable operators were presented and the map which act on matrices of the operators were characterized. Okelo and Aminer [67] completely characterized norms that are bounded, gave norm-convergence in $NA(H)$-classes via the extension of orthogonality. Norm inequalities that are new of matrices of norm-attainable operators were presented and the map which act on matrices of the operators were characterized. Okelo and Aminer [67] completely characterized norms that are bounded, gave norm-convergence in $NA(H)$-classes via the extension of orthogonality.

Lemma 2.43. *[67, Lem. 3.2] If* P_N, $Q_N \in NA(H)$ *are norm-attainable then* $P_N + Q_N$, $P_N - Q_N$ *and* λP_N, $\lambda \in \mathbb{C}$ *are norm-attainable.*

Lemma 2.43 shows that $P_N + Q_N$, $P_N - Q_N$ and λP_N, $\lambda \in \mathbb{C}$ are norm-attainable. This study we have done norm estimates for norm-attainability derivations in Banach algebras.

Theorem 2.44. *[67, Thm. 3.6] An operator* $B \in NA(H)$ *which is nor-mal is norm attainable.*

Theorem 2.44 shows that $B \in NA(H)$ is normal but we have considered conditions of norm-attainability for derivations in Banach algebras.

Okelo [64] considered orthogonal and norm-attainable of operators in Banach spaces, gave in details the generalization of norm-attainability and orthogonality and characterization. The conditions that are sufficient and necessary for norm-attainable operators in a Hilbert space, result on ker-nel of elementary operators and the orthogonal range when implemented by norm-attainable operators in Banach spaces were given.

Proposition 2.45. *[64, Prop. 3.1] Let* $B, C, D \in \Omega$ *with* $DC = 1$ *(1 is an identity element of* Ω*). Then a generalized-derivation* $\delta_{A,B} = BY - YC$ *and an elementary operator* $\Theta_{B,C}(Y) = BYC - Y$, $R_C(Ran\delta_{B,D}) \cap Ker\delta_{B,D} = \overline{Ran\Theta_{B,C}} \cap Ker\Theta_{B,C}$. *Therefore, if* $\overline{Ran\delta_{B,D}} \cap Ker\delta_{B,D} = \{0\}$ *then* $\overline{Ran\Theta_{B,C}} = \cap Ker\Theta_{B,C} = \{0\}$.

Proposition 2.45 investigates the generalized derivation but we have established the both inner and generalized derivation in Banach algebras.

Theorem 2.46. *[64, Thm. 3.10] Let the normal operators be*

$B', C', D', E' \in B'(H)$ *such that* $B'C' = B'C'$, $C'E' = E'C'$, $B'B'^* \leq D'D'^*$, $C'^*C' \leq E'^*E'$. *For an elementary operator* $U'(X') = B'X'C' - D'X'E'$ *and* $T' \in B'(H)$ *satisfying* $B'T'C' = D'T'E'$, $\|U'(X') + T'\| \geq \|T'\|$ *for all* $X' \in B'(H)$.

Theorem 2.46 determines the lower norm estimate for normal operators but we investigated the upper norm-estimates for norm attainable deriva-tions in Banach algebras.

Odero, Agure, Nyamwala [56] showed that the mapping $\Delta_{A'B'} : B(H) \to B(H)$ is a generalized derivation of two bounded operators $A', B' \in B(H)$ induced by A' and B' were defined by $\Delta_{A',B'}(Y) = A'Y - YB'$, therefore, the norm $\|\Delta_{A',B'}\| = \|A'\| + \|B'\|$ for all $A', B' \in B(H)$ was given.

Theorem 2.47. *[56, Thm. 1] Let $J, K \in B(H)$ and $\delta_{JK} : B(H) \to B(H)$. Then $\|\delta_{J,K}\| = \|J\| + \|K\|$ for all $B(H)$.*

Theorem 2.47 determined the norm of generalized derivation operator but we have considered norm attainability conditions for derivations in Banach-algebras.

Theorem 2.48. *[56, Thm. 2] Let the distance from A and B to the scalar multiple of the identity be $\delta(A) = \inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}$ and $\delta(B) = \inf\{\|B - \lambda\| : \lambda \in \mathbb{C}\}$. Then $\|\delta_{AB/B(H)}\| = \|A\| + \|B\|$.*

Theorem 2.48 determined the distance from A and B on a generalized inner derivation operator but we considered norm attainability conditions for derivations in Banach-algebras.

Okelo and Mogotu [59] gave norms of commutators of normal operators for generalized inequalities and established the commutations of derivation for orthogonality and norm inequalities.

Theorem 2.49. *[59, Thm. 3.2] Let the operators $M, N, X \in B(H)$, the pair (M, N) satisfies Fuglede Putnam's property and $D \in ker(\delta)_{M,N}$ where $D \in B(H)$ then $\|\delta_{M,N}X + D\| \geq \|D\|$.*

Theorem 2.49 investigates the lower norm estimate for a generalized derivation for a pair of operators (M, N) but we have determined norm estimate for inner derivation in Banach algebras.

Corollary 2.50. *[59, Cor. 3.3] Let the operators $M, N, X \in B(H)$ and $D \in ker(\delta_{M,N})$ then $\|\delta_{M,N} X + D\| \geq \|D\|$.*

Corollary 2.50 determines the lower norm estimate but we have considered norm estimate for derivations in Banach algebras.

Okelo [60] characterized norm-attainable classes in terms of orthogonality by giving norm-attainability conditions that were necessary and sufficient for Hilbert space operators first and the orthogonality result on the ker-nel and range of norm-attainable classes in elementary-operators when implemented by norm attainable operators was given.

Proposition 2.51. *[60, Prop. 3.2] Let X and Y be norm-attainable Hermitian elements. Then $\delta_{X,Y}$ is also norm-attainable Hermitian.*

Proposition 2.51 gave the condition for generalized derivation for norm-attainable Hermitian elements but we determined norm-attainability condition for derivations in Banach algebras.

Corollary 2.52. *[60, Cor. 3.3] If A and B are norm attainable and normal elements in Ω then $\delta_{A,B}$ is also norm attainable and normal.*

Corollary 2.52 establishes norm-attainability condition of normal elements in Ω for a generalized derivation but we have considered inner derivations in Banach algebras.

Abolfazl [2] determined the norm of the inner Jordan $*$-derivation δ_S :

T $\rightarrow ST - T^*S$ acting on the Banach algebra $B(H)$. It was shown that $\|\delta_S\| \geq 2 \sup_{\lambda \in W_0(S)} |\lambda|$ in which $W_0(S)$ is the maximal-numerical range of operator S. Determined the norm of inner Jordan $*$ derivation δ_S :
T $\rightarrow ST - T^*S$ which act on the Banach algebra $B(H)$. It was shown that $\|\delta_S\| \geq 2 \sup_{\lambda \in W_0(S)} |\lambda|$ in which $W_0(S)$ is the maximal-numerical range of operator S.

Theorem 2.53. *[2, Thm. 2.1] Let H be a Hilbert-space and let* $S \in B(H)$*. If* $\lambda \in W_0(S)$*, then* $\|\delta_S\| \geq 2(\|S\|^2 - |\lambda|^2)^{\frac{1}{2}}$.

Theorem 2.53 estimates the lower norm of a maximal numerical range of operator S but we have determined the norm estimate for norm-attainable derivations in Banach algebras.

Corollary 2.54. *[2, Cor. 2.2] Let H be a Hilbert-space and let* $S \in B(H)$ *then* $\|\delta_S\| = 2\|S\|$ *if and only if* $0 \in W_0(S)$.

Corollary 2.54 determines the norm of a maximal numerical range of operator T but we considered norm estimate for norm-attainable derivations in Banach algebras.

Gyan [28] obtained precisely when zero belongs to maximal numerical range of composition operators on H and then characterized the norm-attainability of derivations on $B(H)$.

Theorem 2.55. *[28, Thm. 1.4] For* $B \in B(H)$, $\|\delta_B\| = 2 \inf \|B - \lambda I\|$: $\lambda \in \mathbb{C}$.

Theorem 2.55 determined the norm of inner derivations but we deter-mined norm estimates for the derivations.

Theorem 2.56. *[28, Thm. 1.5] For $F \in B(H)$,*

$$\|\delta_{F,G}\| = \inf \|F - \lambda I\| + \|G - \lambda I\| : \lambda \in \mathbb{C}.$$

Theorem 2.56 determined the norm of generalized derivations but we have determined norm estimates for the derivations.

Okelo [71] norm-attainability for hyponormal operators that are compact were characterized, sufficient conditions for a compact hyponormal oper-ator that is linear and bounded on an infinite dimension for a complex Hilbert space to be norm attainable were given. Further, the structure and other properties of compact hyponormal operators when they are self-adjoint, normal and norm attainable with their commutators were discussed in general.

Proposition 2.57. *[71, Prop. 3.1] Let $K' \in B(H_1, H_2)$ be compact hyponormal. Then*

(i). $m(K') = m(|K'|)$.

(ii). $m(K') = d'(0, \sigma(|K'|))$.

(iii). $m(K') > 0$ *if and only if* $R(K')$ *is closed and* K' *is one-to one (*K' *is bounded below).*

(iv). in particular if $H_1 = H_2 = H$ *and* $K'^{-1} \in B(H)$*, then* $m(K') = \frac{1}{\|K'^{-1}\|}$.

(v). if $H_1 = H_2 = H$ *and* K' *is normal, then*

(a) $m(K') = d(0, \alpha(K'))$.

(b) $m(K') = m(K'^*)$.

(c) $m(K'^n) = m(K')^n$ *for each* $n \in \mathbb{N}$.

(vi). if $K' \geq 0$, *then* $m(K') = m(K'^{\frac{1}{2}})^2$.

Proposition 2.57 established conditions for compact and hyponormal operators but we have considered norm-attainability conditions for deriva-tions in Banach algebras.

2.3 Norm estimates for derivations

Lumer [46] obtained a sharp estimate not only from $|sp(R)|$ = spectral radius of R but also $|sp(R)|$ in terms of $\sup(|X(R)|, |X(R^n)|^{1/n})$, for an even integer n which is positive. These are

$$|sp(R)| \leq \sqrt{3} \sup(|X(R)|, |X(R^2)|^{1/2})$$

$|sp(R)| \leq \sigma_n \sup(|X(R)|, |X(R^n)|^{1/n})$, $n = 4, 6, 8...$, where $\sigma_n = \sqrt{7}$, generally, σ_n can be calculated as a polynomial root which depend on n. The question about the constants was answered completely for an estimate $\|R\| \leq c_1|X(R)| + c_2|X(R^2)|^{1/2}$ which was expressed as

$\sup \alpha(|X(R)|, \beta|X(R^2)|^{1/2})$ and then compared the estimates. Fur-ther, the aspects of general problem were discussed and then gave ap-plications by introducing an invariant $\delta(C)$ defined for all unital Banach algebra C.

Theorem 2.58. *[46, Thm. 1] There* $\exists$ *constants* c_1, c_2 *such that for any real Banach-space* Y, *one has* $\|A\| \leq c_1|W'(A)| + c_2|W'(A^2)|^{1/2}$, $\forall A \in$

$B(Y)$.

Theorem 2.58 discussed the upper norm estimate but we have considered lower norm estimates for derivations in Banach algebras.

Archbold [1] investigated whether the simple triangle inequality $\|T(a, A)\| \leq 2t(a, Z)$ if applied holds. $D(A)$ was defined to be a minimum value D in $[0, \infty]$ so that $t(a, Z) \leq D\|T(a, A)\|$. The behaviour of D in ideals and quotients were discussed which proved that $D_S(A) \leq 1$ for a weakly cen-tral C^*-algebra A and considered a class of n-homogeneous C^*-algebras that are special. D and D_S was investigated and approximated finite-dimension $(AF)C^*$-algebra in that context and an example was given to show certain estimates.

Proposition 2.59. *[1, Thm. 4.1] Let P be an ideal of a C^* algebra U. Then $T(P) \leq 2T(U)$.*

Theorem 2.59 determined the upper norm estimate but we have estab-lished the norm estimate for norm-attainable derivations in Banach alge-bras.

Fong [25] considered $\lambda(M)$ defined as the smallest number $\|Z\|^2$ of Z that satisfy $[Z^*, Z] = M$ and showed that $1 \leq \lambda(M) \leq 2$ and $\lambda(M)$, M was suitably chosen if it is close to 2.

Proposition 2.60. *[25, Prop. 1] If $[S^*, S] = T$, then $\|S\|^2 \geq \|T\|$.*

Proposition 2.60 established the lower estimates for operators that are self-adjoint. In this study we have determined lower-estimates for norm attainable derivations in Banach-algebras.

Proposition 2.61. *[25, Prop. 2] If K is a self-adjoint matrix with $trK = 0$, then there exists some matrix $W^{'}$ such that $[W^{'*}, W^{'}] = K$ and $\|W^{'}\|^2 \leq 2\|K\|$.*

Proposition 2.61 investigated the upper estimates for self-adjoint opera-tor. In this study we have determined upper-estimates for norm attainable derivations in Banach-algebras.

Matej [47] estimated the distance of $d_1 d_2$ to the generalized derivations and the normed algebra of $M^{'}$ and considered the cases when $M^{'}$ is an ultraprime, when $d_1 = d_2$ and $M^{'}$ are ultrasemiprime and when a Von Neumann algebra is $M^{'}$ from the equation $\|M^{'} + R^{'}\| = \|M^{'}\| + \|R^{'}\|$, $M^{'}, R^{'} \in B(H)$.

Theorem 2.62. *[47, Thm. 1] Let E be an ultraprime normed-algebra, and let $d_1, d_2 \in \delta_b(E)$. If a constant $c > 0$ satisfies, then $dist(d_1 d_2, \delta_b(E)) \geq (c^2/6)\|d_1\|\|d_2\|$.*

Theorem 2.62 discussed the lower estimate for ultraprime normed algebra and we have considered lower norm estimate for derivations in Banach algebras.

Theorem 2.63. *[47, Thm. 3] Let A be a Von-Neumann algebra. If $d_1, d_2 \in \delta_b(A)$, then $dist(d_1 d_2, \delta_b(A)) \leq (1/2)\|d_1\|\|d_2\|$. For every*

$$dist(d^2, \delta_b(A)) = (1/2)\|d\|^2.$$

Theorem 2.63 discussed the upper estimate for a Von Neumann algebra and we have considered upper norm estimate for derivations in Banach algebras.

Baxter [6] provided the supremum on $\|B^{-1}\|_2$ when the points $(y_i)^n_1$ form a subset of the integer L^d, and a conditional definite negative function ϕ of order 1, which included the multi-quadric for functions set that are large. Further, a constructive proof was provided that a minimum bound is not valid and a relevant method to analyze the problem on estimation of eigenvalues such an interpolation matrix was commented on.

Theorem 2.64. *[6, Thm. 4.1] Let $B \in R^{nxn}$ be a symmetric matrix with eigenvalues $\alpha_1 \geq ... \geq \alpha_n$. Let D be any subspace of R^n of dimension m. Then we have the inequality* $\max\{s^T Bs : s^T s = 1, s \perp D\} \geq \alpha_{m+1}$.

Theorem 2.64 established the lower estimate of symmetric matrix with eigenvalues but we have determined the lower norm estimate for deriva-tions in Banach algebras.

Kittaneh [40] established the orthogonality, kernel and the range of a normal derivation with its association to operators of norm ideals. Results relating to orthogonality of some derivation that are not normal were obtained.

Theorem 2.65. *[40, Thm. 1] Let $M \in B(H)$ be normal, $S \in M'$, and $T \in B(H)$. If $\delta_M(T) + S \in K_{|\|.\||}$, then $S \in K_{|\|.\||}$ and $|\delta_M(T) + S| \geq |S|$.*

Theorem 2.65 investigates the lower norm estimate for an inner derivation but we have determined for a generalized derivation in Banach algebras.

Corollary 2.66. *[40, Cor. 1] Let P, Q, R $\in B(H)$ such that P and Q are normal and $PR = RQ$. If $Y \in B(H)$ such that $\delta_{P,Q}(Y) + S \in K|\,.\,|$, then $S \in K_{|\|.\||}$ and $|\delta_{P,Q}(Y) + S| \geq |S|$.*

Corollary 2.66 discussed the lower norm estimate for a generalized derivation but we have determined for an inner derivation in Banach algebras.

Stacho and Zalar [89] established the lower estimates for elementary operators of Jordan type in standard Banach algebras.

Proposition 2.67. *[89, Prop. 2] The estimate* $\|U_{a,b}\| \geq \|a\|.\|b\| + |\, a, b\,|$ *holds.*

Proposition 2.67 established the lower norm estimate for a generalized derivation for elementary operators but we determined the norm estimate for an inner derivation in Banach algebras.

Theorem 2.68. *[89, Thm. 4] Let A be a standard operator algebra which acts on a Hilbert space H. If c, d* $\in A$*, then the uniform estimate* $\|U_{c,d}\| \geq 2(\sqrt{2} - 1)\|c\|.\|d\|$ *holds.*

Theorem 2.68 considered the uniform estimate for a generalized derivation in a standard Banach algebra but we have investigated norm estimate for an inner derivation in Banach algebras.

Danko [20] [20] established that for all unitarily invariant norms and for bounded Hilbert space operators there holds $\||C-D|^q|\| \leq 2^{q-1}\||C|C|^{q-1} - D|D|^{q-1}|\|$, $q \geq 2$, if in addition, C and D are self-adjoint then $||||CX + XD|^q|||$ $\leq 2^{q-1}||X||^{q-1}||||A|^{q-1}CX + XD|D|^{q-1}|||$, for all real $q \geq 3$.

Theorem 2.69. *[20, Thm. 3.1] If X and some self-adjoint C and D are in* $B(H)$*, then* $|\||CX + XD|^p\|| \leq 2^{p-1}\|X\|^{p-1}|\||C|^{p-1}CX + XD|D|^{p-1}\||$ *for all real* $p \geq 3$ *and for all unitarily invariant norms* $|\|.\||$.

Theorem 2.69 determined the upper bound for unitarily invariant norms but we have investigated the lower bound for norm-attainable derivations in Banach algebras.

Shinji [86] established that for a holomorphic functions f with $Re\{gf'(g)\} > \alpha$ and $Re\{gf''(g)/f'(g)\} > \alpha - 1$, $(0 \leq \alpha < 1)$ respectively in $\{|g| < 1\}$, estimates of $\sup_{|g|<1}(1-|g|^2)|f''(g)/f'(g)|$ were given and functions Gelfer-convex of exponential order α, β was also considered.

Theorem 2.70. *[86, Thm. 3] Let* $-\infty < \beta < +\infty$, $0 \leq \alpha \leq 1$ *and* $\gamma \geq 0$. *Then for* $f \in K_G(\beta, \alpha, \gamma)$ *we have* $\|f\| \leq |1 - \beta| M(\alpha) + 2\gamma$.

Theorem 2.70 determines the upper norm estimate for holomorphic functions. In this study we have determined norm estimates for derivations in Banach algebras.

Milos, Dragoljub [52] considered elementary operators $x \longrightarrow \sum_{j=1}^{n} v_j x w_j$ that acts on a Banach algebra, v_j and w_j denotes separate generalized scalar elements of commuting families. The ascent estimation and lower bound estimation of an operator was given. Additionally, Fuglede-Putnam theorem for elementary operator is a weak variant with v_j and w_j are strongly commuting families were given i.e $v_j = v_j' + iv_j''$ $(w_j = w_j' + w_j'')$, for all v_j' and v_j'' (w_j and w_j'') commutes. Further, result concerning L^1 estimate in Fourier transform of a class C_{cpt}^{∞} function in R^{2n} was obtained.

Lemma 2.71. *[52, Lem. 2.2] Let* $T \subseteq R^{2n}$ *be a set of balanced Hausdorff dimension c. Then for all* $\delta > 0$ *there exist open set* $U_\delta \supset T$, *such that* $mU_\delta \leq C(T, n)\delta^{2n-c}$ *and* $distT, U_\delta^C \geq \delta/P$.

Lemma 2.71 determined the lower estimate for a general derivation and we have discussed the lower estimate for inner and generalized derivations in Banach algebras.

Barraa and Boumazgour [7] characterized that the norm of bounded operators more than one in a Hilbert-space is the same summation of the norms which showed that $\delta_{S,A,B}$ is convexoid with the convex hull of its spectrum if and only if A and B are convexoid.

Theorem 2.72. *[7, Thm. 2.1] Let $X, Y \in B(H)$ be non zero. Then the equation $\|X+Y\| = \|X\|+\|Y\|$ holds if and only if $\|X'\|\|Y\| \in W(X^*Y)$.*

Theorem 2.72 discussed the operators of norm in a Hilbert space. In our study we have determined lower and upper norm estimate in norm-attainable derivations in Banach algebras.

Corollary 2.73. *[7, Cor. 2.3] Let $Y, Z \in B(H)$ be non zero. If $\|Y\|\|Z\| \in W(Y^*Z)$, then $0 \in \sigma_{ap}(\|Z\|Y - \|Y\|Z)$. The converse holds if any one of Y or Z is an isometric operator.*

Corollary 2.73 established the closure of numerical range of bounded operators. In our study condition for norm-attainability of derivations in Banach algebras has determined.

Richard [78] established the CB-norms of elementary operators and the lower bounds for norms on $B(H)$. The result was concerned with the operator $U_{A,B}X = AXB + BXA$ which showed that $\|U_{A,B}\| \geq \|A\|\|B\|$ which proved a conjecture of Mathieu, other results and formula of $\|U_{A,B}\|_{CB}$ and $\|U_{A,B}\|$ were established.

Theorem 2.74. *[78, Thm. 2] Assume that H is two-dimensional and $D, E \in B(H)$. Let $U_{D,E}(X) = DXE+EXD$. Then $\|U_{D,E}\|_{CB} \geq \|D\|_2\|E\|_2$.*

Theorem 2.74 determined the lower bound for Jordan elementary oper-ator. In this study we have determined lower norm estimates in norm-attainable derivations in Banach algebras.

Theorem 2.75. *[78, Thm. 6] If $C, D \in B(H)$ and $U_{C,D}(X) = CXD + DXC$. Then $\|U_{C,D}\| \geq \|C\|\|D\|$.*

Theorem 2.75 determined the lower bound for Jordan elementary opera-tor. In this study we determined lower norm-estimates in norm attainable derivatives in Banach-algebras.

Richard [79] provided the estimation on the norm of elementary operators that are completely bounded was a direct proof which was possible in $B(H)$ through a generalized theorem by Stampfli [87] and it was shown that an operator J of length l equals to m-norm and $m = l$.

Theorem 2.76. *[79, Thm. 4.3] If $k \geq 1$ and A is a continuous trace C^{*}-algebra which is not K subhomogeneous, then there exists an elementary operator $T \in El(A)$, $T(x) = \sum_{i=1}^{k+1} f_i x g_i$, $f_i, g_i \in A$ for $1 \leq i \leq k+1$ with $\|T\|_k < \|\|_{cb}$.*

Theorem 2.76 determined the upper norm estimate for elementary op-erator but we have considered the norm-estimates for norm attainable derivations in Banach-algebras.

Seddik [81] proved that lower estimate bound $\|T_{M,N}\| \geq 2(\sqrt{2}-1)\|M\|\|N\|$ holds, if it satisfies one of the conditions: (i). A standard operator alge-bra on $B(H)$ is L and $M, N \in L$, (ii). L is a norm ideal on $B(H)$ and $M, N \in B(H)$.

Lemma 2.77. *[81, Lem. 1] We have the following properties:*

(i). $\|U_{J,C,D}\| \geq \sup\{|\langle Ca, y\rangle\langle Dx, v\rangle + \langle Da, y\rangle\langle Cx, v\rangle| : \|a\| = \|y\| = \|x\| = \|v\| = 1\}$.

(ii). $\|U_{J,C,D}\| \geq 2w(C^*D)$.

Lemma 2.77 gave the norm and lower norm estimates for maximal numerical range of operators C, D but we have established norm attainability conditions for derivatives in Banach algebras.

Theorem 2.78. *[81, Thm. 4] We have the following property:*

$$\|U_{J,A,B}\| \geq 2(\sqrt{2} - 1)\|A\|\|B\|.$$

Theorem 2.78 investigates the lower bound in a standard operator algebra. This study we have determined lower and upper norm estimate in norm-attainable derivations in Banach algebras.

Florin, Alexandra [26] estimated the norm of operator $H_{\theta,\lambda} = U_\theta + U_\theta^* + (\lambda/2)(V_\theta + V_\theta^*)$ which is an element on a C^*-algebra $A_\theta = C^*(U_\theta, V_\theta$ uni-taries $: U_\theta V_\theta = e^{2\pi i\theta}V_\theta U_\theta)$. Further, proved for every $\lambda \in \mathbb{C}$ and $\theta \in [\frac{1}{4}, \frac{1}{2}]$ the inequality $\|H_{\theta,\lambda}\| \leq \sqrt{4+\lambda^2-(1-\frac{1}{\tan\theta,\lambda})(1-\sqrt{\frac{1+\cos^2 4\pi\theta}{2}})min\{4, \lambda^2\}}$. This improved the significance of the inequality $\|H_{\theta,2}\| \leq 2\sqrt{2}, \theta \in [\frac{1}{4}, \frac{1}{2}]$,

Lemma 2.79. *[26, Lem. 2.2] For every* $\theta \in [0, \frac{1}{2}]$, $|\sum_m X_m X_{m-1} \cos(2m-1)\pi\theta| \leq \sqrt{\frac{1+|\cos 2\pi\theta|}{2}}$.

Lemma 2.79 determined the upper estimate for almost Mathieu operators but we have established the upper estimate for norm-attainable deriva-tions.

Lemma 2.80. *[26, Lem. 3.1] If* $(Y_m)_{m\in\mathbb{Z}_q}$ *is a unit vector in* $l^2(\mathbb{Z}_q)$, *then*

$$2\sum_m C_m^2 Y_m^2 + \sum_m Y_{m+1}Y_{m-1} \leq 1 + \sqrt{2(1 + \cos^2 4\pi\theta)}.$$

Lemma 2.80 established the upper estimate for a unit vector. In this study we have determined upper norm-estimates for norm attainable derivations in Banach-algebras.

Man-Duen, Chi-kwong [50] showed that triangle inequality served an up-per norm bound of an ultimate estimate for the sum operators that is sup$\{ \|T^*RT + V^*SV\| : T$ and $V \}$ are unitaries

$= \min \|R + \lambda I\| + \|S - \lambda I\| : \lambda \in \mathbb{C}$. The result discussed had rela-tionship to normal dilations, spectral sets and the Von Neumann inequal-

ity.

Corollary 2.81. *[50, Cor 3.2] Let $P, Q \in B(H)$. Then $\|P + Q\| \leq \sup\{ \|U^*P U + V^*QV\| : U$ andV areunitaries$\}$. The equality holds if and only if there exists $\mu_0 \in \mathbb{C}$, such that $\|P + Q\| = \|P + \mu_0 I\| + \|Q - \mu_0 I\|$.*

Corollary 2.81 determined norm of the normal operators but we have determined upper and lower norm estimates for derivations.

Gil [29] considered commuting matrices of matrix valued analytic function and established a norm estimate, in particular, two matrices of matrix valued functions on a tensor product in a Euclidean space were explored.

Theorem 2.82. *[29, Thm. 1.1] Let S and T be commuting n x n-matrices and $f(z, w)$ be regular on a neighborhood of $co(S)$ X $co(T)$.*

$$\text{Then } \|f(S, T)\| \leq \sum_{j,k=0}^{j+k\leq n-1} \eta_j \eta_k g^j(T) \sup_{z\in co(s), w\in co(T)} |f^{j,k}(z, w)|.$$

Theorem 2.82 considered the norm estimate for commuting matrices but we have determined upper and lower norm-estimates for norm attainable derivations in Banach-algebras.

Yong, Toshiyuki [95] gave a norm estimate on pre-schwarzian derivatives of a specific type of convex functions by introducing a maximal operator of independent interest of a given kind. The relationship between the convex functions and the Hardy spaces was discussed.

Theorem 2.83. *[95, Thm. 4.3] Let* $-1 \leq N < M \leq 1$. *If* $f \in K_{(\varphi M,N)}$, *then* $\|T_f\| \leq \frac{2(M-N)}{1+\sqrt{1-N^2}}$, *and equality holds when* $f = K_{\varphi M,A}$.

Theorem 2.83 established the upper estimate for the univalent functions. In this study we have determined lower and upper norm-estimates for norm attainable derivations in Banach-algebras.

Corollary 2.84. *[95, Cor. 4.5] Let* $0 \leq t \leq 1$, *functions* $f \in S_{\varphi}^{*}{}_{-t,t}$ *satisfy the inequality* $\|T_f\| \leq \frac{4t}{1+\sqrt{1-t^2}} + 2t$.

Corollary 2.84 established the upper estimate for the univalent functions. In this study we have determined lower and upper norm estimates for norm-attainable derivations in Banach algebras.

Bonyo and Agure [10] characterized the norm ideal on norm of inner derivation to be equal to the quotient algebra and investigated them when the normal and hyponormal operators are implementing them on norm ideals.

Theorem 2.85. *[10, Thm. 2.1] Let J be a norm ideal in* $B(H)$ *and* $B \in B(H)$. *Then* $\|\delta_{[B]}|B(H)/J\| \leq 2d(B)$.

Theorem 2.85 determined the upper norm estimate for an inner derivation but we have established norm estimate for a generalized derivation in Banach algebras.

Theorem 2.86. *[10, Thm. 2.5] Let* $B(H)$ *be the algebra of bounded linear operators on a Hilbert space H, J is a primitive norm ideal in* $B(H)$. *Then for an S-universal operator* $B \in B(H)$, $\|\delta_{[B]}|B(H)/J\| = \|\delta_B|J\|$.

Theorem 2.86 determined the norm of inner derivation but we have considered norm of generalized derivation in Banach algebras.

Bonyo and Agure [11] investigated the relation between the inner deriva-tion implemented by Z on norm J and the numerical-range of an operator

$Z \in B(H)$ with its diameter and considered application of T -universality on the relation.

Theorem 2.87. *[11, Thm. 2.3] For any operator* $X \in B(H)$ *and each norm ideal J in* $B(H)$, *diameter* $(W(X)) \leq \|\delta_X |J\|$.

Theorem 2.87 discussed the upper norm estimate for a the diameter of numerical range in a norm ideal but we have established norm-estimates in norm attainable derivations in Banach-algebras.

Okelo, Okongo and Nyakiti [58] investigated the project tensor-product, $V_\Gamma{}' \otimes_\rho W_\Gamma{}'$ of these algebras. It was established that $\|\Delta_{S'}\| \leq \|\Delta^{(1)}{}_{S'} + \Delta^{(2)}{}_{S'}\| \leq 2\|\Delta_{S'}\|$ holds if $\lambda = \sum_i v_i{}' \otimes w_i{}'$ belongs to $A_\Gamma \otimes_\rho B_\Gamma$ and $\Delta_{S'}$ on
λ is a norm-attainable α-derivation given by $\Delta_{S'} = \Delta^{(1)}{}_{S'} + \Delta^{(2)}{}_{S'}$.

Lemma 2.88. *[58, Lem. 4.2] If* $\delta_N{}^{(1)}$, $\delta_N{}^{(2)}$ *are norm-attainable* α *and* α'*-derivations respectively, then* $\delta_N = \delta_N{}^{(1)} + \delta_N{}^{(2)}$ *is norm-attainable.*

Lemma 2.88 shows the sum of $\delta_N{}^{(1)}$, $\delta_N{}^{(2)}$ inner derivations are norm-attainable but we have done norm-attainability for a generalized deriva-tion in Banach algebras.

Theorem 2.89. *[58, Thm. 4.3] Let* δ_N *,* $\delta_N^{(1)}$ *and* $\delta_N^{(2)}$ *be norm attainable* α*,* α' *and* α''*-derivations respectively where* $\|\delta_N\| = \|\delta_N^{(1)}\| + \|\delta_N^{(2)}\|$*, then* $\|\delta_N\| \leq \|\delta_N^{(1)}\| + \|\delta_N^{(2)}\| \leq 2\|\delta_N\|$.

Theorem 2.89 investigates the norm and upper norm estimate for inner derivations but we have discussed norm of a generalized derivations in Banach algebras.

Bonyo and Agure [9] gave the definition of inner derivations implemented by A, B respectively on $B(H)$ as $\delta_A(Y) = AY - YA$ $\delta_B(Y) = BY - YB$ and generalized derivation by $\delta_{A,B}(Y) = AY - YB$ $\forall Y \in B(H)$. Further, a relation between the norms of δ_A, δ_B and $\delta_{A,B}$ on $B(H)$ was specifically established when the operators A, B are S-universal.

Theorem 2.90. *[9, Thm. 3.1] If* $C \in B(H)$ *is S-universal, then* $\|\delta_C|B(H)\| = 2\|C\|$.

Theorem 2.90 determines the norm of inner derivation but we have considered the norm-attainable conditions in Banach algebras.

Theorem 2.91. *[9, Thm. 3.2] Let* $C, D \in B(H)$ *be S-universal. Then*

$$\|\delta_{C,D}|B(H)\| \leq \frac{1}{2}(\|\delta_C|B(H)\| + \|\delta_D|B(H)\|).$$

Theorem 2.91 investigates the upper norm estimate for an inner and derivations. This study we have determined upper and lower estimates for norm attainable derivatives in Banach-algebras.

Pablo, Jussi [75] provided theoretic estimate of two functions for the essential norm as a composition operator C_φ that acts on the space $BMOA$

(bounded mean oscillation for analytic functions); one in terms of the n-th power φ^n denoted by φ and the other involved the Nevanlinna counting function.

Lemma 2.92. *[75, Lem. 2.3] Let φ be an analytic self-map of* K. *Then*

$$\limsup_{|\varphi(b)|\to 1} \|\sigma_{\varphi(b)} \circ \varphi\|_{*,2} \leq 2 \limsup_{n\to\infty} \|\varphi_n\| \text{ and}$$

$$\limsup_{|\varphi(b)|\to 1} \|\sigma_{\varphi(b)} \circ \varphi\|_{\beta} \leq 2 \limsup_{n\to\infty} \|\varphi_n\|_{\beta}.$$

Lemma 2.92 determined the upper estimate for analytic self-map but we have discussed the upper norm estimates for derivations in Banach algebras.

Lemma 2.93. *[75, Lem. 3.2] We have*

$$\limsup_{|\varphi(a)|\to 1} \|\varphi_a\|_2 \leq \|C_\varphi\|_{e,BMOA}.$$

Lemma 2.93 discussed the upper estimate composition of operators. In this study we have determined the norm-estimates for norm attainable derivations in Banach-algebras.

Kingangi, Agure and Nyamwala [43] attempted the result on lower bound of the norms for finite dimensional operators.

Theorem 2.94. *[43, Thm. 2.2] Let $U_{A,S}$ be the Jordan-elementary opera-tor with $A, S \in B(H)$ fixed, and with $S \neq 0$. Then $\|U_{A,S}\| \geq \sup_{\lambda\in W_S(A^*,S)}\{ \|S\|A + \bar{\lambda}\|S^*\|S \}$, where $W_S(A^*, S)$ is the maximal numerical range of A^*, S relative to S, A^* is the Hilbert adjoint of S.*

Theorem 2.94 established the lower estimate on the maximal numerical range of operator but we considered lower estimate for norm-attainable derivations in Banach algebras.

Corollary 2.95. *[43, Cor. 2.3] Let H be a complex Hilbert space and* $X^{'}$, $Y^{'}$ *be bounded linear operators on H. Let* $0 \in W_{Y'}(X^{'*}, S) \cup W_{X'}(Y^{'*}, X^{'})$. *Then we have* $U_{X',Y'} \geq \|X^{'}\|\|Y^{'}\|$.

Corollary 2.95 determined the lower estimate on Jordan elementary operators but we have determined lower norm-estimates for norm attainable derivations in Banach algebras.

Odero, Agure, Rao [57] determined the norm of symmetric operator in an algebra which is two-sided. More precisely, investigated the injection of tensor norm through the lower bound of the operator. In addition, the irreducible C^*-algebra on the inner derivation norm was determined and Stampfli [87] confirmed the result for these algebras.

Theorem 2.96. *[57, Lem. 3.3] Let* $\mu \in W(T)$. *Then* $\|\delta_T\| \geq 2(\|T\|^2 - |\mu|^2)^{1/2}$.

Lemma 2.96 determined the lower estimate for a derivation but we have considered upper estimate for derivation in Banach algebras.

Theorem 2.97. *[57, Thm. 3.4]* $\|\delta_S\| = 2\|S\|$ *if and only if* $0 \in W(S)$.

Theorem 2.97 established the norm of a derivation on a numerical range but we have determined the norm estimate for the derivation in Banach algebras.

Kinyanjui [41] estimated the norm-attainability for elementary opera-tor on inner-derivation, generalized-derivation, basic-elementary operator and Jordan-elementary operator under norms.

Theorem 2.98. *[41, Thm. 2.3] Let H be an infinite dimensional complex nonseparable Hilbert space and* $\varepsilon[NA(H)]$ *be the set of all norm-attainable operators. Let* $M_{S,T} \in \varepsilon[NA(H)]$ *and* $X \in H$ *be defined by* $M_{S,T} = SXT$ *then,* $\|M_{S,T}(X)\| = \|S\|\|T\|$.

Theorem 2.98 discussed norm-attainability for basic elementary operators under but we have determined upper and lower norm-estimates for norm attainable derivations in Banach-algebras.

Wafula, Okelo and Ongati [93] studied normally represented operator which is a special type of elementary operator and results showed that elementary equals its largest single value that is $U_i(M) = \|M\|$ since

$S_{A,B} = A \quad hB + B \quad hA$ is represented normally, then $\|S_{A,B}\|_{Inj} \geq$

$\sqrt{\ } \_ \quad A \quad B. \quad \otimes$

$- \quad \| \quad \|\|\| \; \|$

Proposition 2.99. *[93, Prop. 4.13] Let H be a complex-Hilbert space and*

$M : B(H) \rightarrow B(H)$ *be a basic elementary operator. Then* $S_i(M) = \|M\|$ *such that* $S_i(M)$ *are singular values of M.*

Proposition 2.99 found the norm-attainability for basic elementary oper-ator but we have established norm-attainability for derivations in Banach algebras.

Theorem 2.100. *[93, Thm. 4.14] Let* $U_{A,B}(Y) = AYB + BYA$ *be nor-mally represented then,* $U_{A,B}\|_{CB} \geq \|A\|\|B\|$ *for* $A, B \in B(H)$.

Theorem 2.100 investigated the lower norm estimate for a generalized derivation but we have considered norm estimate for inner derivation in Banach algebras.

Elena, Lorenza, Ivan [24] studied properties of continuity of module spaces for operators of $\imath$-pseudo-differential $Op_\imath(c)$ in a Wiener amalgan space with a symbols c and obtained a bounded result for $\imath \in (0, 1)$ where $\imath = 0$ and $\imath = 1$ at end points and other operators were unbounded. In addition, it was exhibited the operator norm for the function $\imath \in (0, 1)$ has an upper bound which is independent on parameter $\imath \in (0, 1)$ was found.

Proposition 2.101. *[24, Prop. 4.2] Let $m \in M_v(\mathbb{R}^{2d})$ then $a \in X(FL^2{}_v, L^2)(\mathbb{R}^{2d})$ and $\tau \in (0, 1)$. Then the operator $Op_\tau(b)$ is bounded on $M_m{}^2$ with $\|Op_\tau(b)f\|_{M_m^2} \leq C\|b\|_{FL^2_v,L^2} \|f\|_{M_m^2}$, where the constant $C > 0$ is independent of τ.*

Proposition [2.101] established the upper bound τ-pseudo-differential operators but we have considered norm-estimates for norm attainable derivations in Banach algebras.

From the above literature review it is very clear that norm-attainability for elementary operators have been done thus we have determined norms of derivations as an example of elementary-operators when they are implemented by operators that are norm attainable and we have estimated the norms of derivations in Banach algebras.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

The research methods involves the use of known inequalities like Cauchy-Schwarz inequality, triangle inequality, H¨olders inequality, Bessel's inequality. Technical approaches like direct sum, polar decomposition and tensor product were useful to our work.

3.2 Known inequalities

3.2.1 Cauchy-Schwarz inequality

Let the inner product space be S and $\|s\| = \sqrt{\langle s, s \rangle}$ $\forall s' \in S'$ then $|\langle s', t' \rangle| \leq \|s'\|\|t'\|$ $\forall s', t' \in S'$. Indeed, if $t' = 0$ and $\langle s', 0 \rangle = 0$ then the equality is satisfied. Let $t' = 0$. For any scalar α, 0

$\leq \|$ $-$ $\|$ $-$

$\alpha t', s' - \alpha t' \quad s', s' - \alpha\overline{s', t'} - \alpha[t', s' - \alpha t', \overline{t'}]$, let $\alpha[t', s' - \alpha t', t'] = 0$.

If we choose $\alpha = \frac{t', s'}{t', t'}$, then we have

$$
\begin{aligned}
0 &\le s', s' - \frac{t', s'}{t', t'} s', t' \\
&\le \|s'\|^2 - \frac{|s', t'|^2}{\|t'\|^2} \quad (\textit{multiply by } \|t'\|^2) \\
&\le \|s'\|^2 \|t'\|^2 - |s', t'|^2 \\
&\le \|s'\|^2 \|t'\|^2 .
\end{aligned}
$$

Taking positive square roots it yields $|s', t'| \le \|s'\|\|t'\|$.

Cauchy-Schwarz inequality will be used to determine the upper norm-estimates for norm attainable derivations in Banach-algebras.

3.2.2 Triangle inequality

$\forall s, t \in S$, $\|s + t\| \le \|s\| + \|t\|$. Indeed $\|s + t\|^2 = s + t, s + t$

$$
\begin{aligned}
\|s + t\|^2 &= s + t, s + t \\
&= s, s + s, t + t, s + t, t \\
&= s, s + s, t + s, t + \overline{t, t} \\
&= s, s + 2Re\, s, t + t, t \\
&\le s, s + 2|s, t| + t, t \\
&= \|s\|^2 + 2\|s\|\|t\| + \|t\|^2 \\
&\le (\|s\| + \|t\|)^2
\end{aligned}
$$

Taking positive square root it yields $\|s + t\| \le \|s\| + \|t\|$.

Triangle inequality will be used to determine the upper norm-estimates for norm attainable derivations in Banach-algebras.

3.2.3 Hölder's Inequality

Let $s_n \in l_x$ and $t_n \in l_y$ where $x > 1$ and $1/x+1/y = 1$, then $\sum_{k=1}^{\infty} |t_k s_k| \leq (\sum_{k=1}^{\infty} |s_k|^x)^{1/x}(\sum_{k=1}^{\infty} |t_k|^y)^{1/y}$.

Indeed if $\sum_{k=1}^{\infty} |s_k|^x = 0$ or $\sum_{k=1}^{\infty} |t_k|^y = 0$ then the inequality holds.

Assume $\sum_{k=1}^{\infty} |s_k|^x \neq 0$ and $\sum_{k=1}^{\infty} |t_k|^y \neq 0$ then $k = 1, 2, ..$ by Young's inequality then

$$\frac{|s_k||t_k|}{(\sum_{k=1}^{\infty} |s_k|^x)^{1/x}(\sum_{k=1}^{\infty} |t_k|^y)^{1/y}} \leq 1/x \frac{|s_k|^x}{\sum_{k=1}^{\infty} |s_k|^x} + 1/y \frac{|t_k|^y}{\sum_{k=1}^{\infty} |t_k|^y}$$

hence

$$\frac{\sum_{k=1}^{\infty} |s_k t_k|}{(\sum_{k=1}^{\infty} |s_k|^x)^{1/x}(\sum_{k=1}^{\infty} |t_k|^y)^{1/y}} \leq 1/x + 1/y = 1$$

$$\Rightarrow \sum_{k=1}^{\infty} |s_k t_k| \leq (\sum_{k=1}^{\infty} |s_k|^x)^{1/x}(\sum_{k=1}^{\infty} |t_k|^y)^{1/y}.$$

Hölder's Inequality will be of significance in the determination of the upper norm-estimates for norm attainable derivations in Banach-algebras.

3.2.4 Bessel's inequality

Let $[v_i]_{i=1}^{\infty}$ be an orthonormal set in an inner product space D then for an arbitrary $d \in D$, $\sum_{i=1}^{\infty} |\langle d, v_i\rangle|^2 \leq \|d\|^2$. Indeed we are supposed to show that $0 \le \|d - \sum_{i=1}^{m} \langle d, v_i\rangle v_i\|^2$. Let $\alpha_i = \langle d, v_i\rangle$

$$
\begin{aligned}
0 &\le \Big\|d - \sum_{i=1}^{m} \alpha_i v_i\Big\|^2 \\
&\le \Big\langle d - \sum_{i=1}^{m} \alpha_i v_i,\ d - \sum_{i=1}^{m} \alpha_i v_i \Big\rangle \\
&\le \langle d, d\rangle - \Big\langle d, \sum_{i=1}^{m} \alpha_i v_i\Big\rangle - \Big\langle \sum_{i=1}^{m} \alpha_i v_i, x\Big\rangle + \Big\langle \sum_{i=1}^{m} \alpha_i v_i, \sum_{i=1}^{m} \alpha_i v_i\Big\rangle \\
&\le \|d\|^2 - \sum_{i=1}^{m} \overline{\alpha_i} \langle d, v_i\rangle - \sum_{i=1}^{m} \alpha_i \langle v_i, x\rangle + \sum_{i=1}^{m}\sum_{i=1}^{m} \alpha_i \overline{\alpha_i} \langle v_i, v_i\rangle \langle v_i, v_i\rangle \\
&\le \|d\|^2 - \sum_{i=1}^{m} \overline{\alpha_i}\alpha_i + \sum_{i=1}^{m} \alpha_i \overline{\alpha_i} + \sum_{i=1}^{m} |\alpha_i|^2 \\
&\le \|d\|^2 - \sum_{i=1}^{m} |\alpha_i|^2
\end{aligned}
$$

$\sum_{i=1}^{m} |\alpha_i|^2 \le \|d\|^2$.

Bessel's inequality will be used to determine the upper norm-estimates for norm attainable derivations in Banach-algebras.

3.3 Technical approaches

In this section technical approach such as tensor product was used to solve the problem stated. We employed polar decomposition and direct sum decomposition in determining norm-attainability conditions and norm estimates for derivations.

3.3.1 Tensor product

If S and T are vector spaces over K, let M be the subspace of the vector space K_{SxT} then the vectors $\alpha(s, t) + \beta(s', t) - (\alpha s + \beta s', t)$ and $\alpha(s, t) + \beta(s, t') - (s, \alpha t + \beta t')$ $\forall \alpha, \beta \in K$ and $s, s' \in S$ and $t, t' \in T$ are generated. Then the space of the quotient K_{SxT}/M is the tensor product of S and T which is denoted by $S \otimes T$.

The technical approach was useful in determining norm estimates.

3.3.2 Direct sum decomposition

A vector space Y is a direct sum of two subspaces A' and B' of Y written as $Y = A' \oplus B'$, if each $y \in Y$ is uniquely represented by $y = a' + b'$, $a' \in A'$, $b' \in B'$.

Direct sum decomposition was used to prove that various derivations are norm-attainable.

3.3.3 Polar decomposition

Let $X \in B(H)$, then a partially isometry W exists with initial space $R(X^*)$ and final space $R(X)$ such that $X = W\,(X^*X')^{1/2} = (XX^*)^{1/2}W$.

Polar decomposition has been used to find square roots of operators on Lemmas and theorems in our proofs.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, we give results obtained on norm-attainability conditions for derivations, upper norm estimates for derivations and lower norm estimates for derivations in Banach algebras.

4.2 Norm-attainability conditions

In this section, we give results on norm-attainability conditions for derivations. We begin with the following proposition.

Proposition 4.1. *Let H be a complex Hilbert space and $B^{'}(H)$ the algebra of all bounded linear operators on H. $A^{'} \in B^{'}(H)$ is norm-attainable if and only if its adjoint $A^{'*} \in B^{'}(H)$ is norm-attainable.*

Proof. Given $A^{'} \in B^{'}(H)$ is norm-attainable then we need to show that $A^{'*} \in B^{'}(H)$ is norm-attainable. If $A^{'} \in B^{'}(H)$ is norm attainable then by definition of norm-attainability there exist a unit vector $x^{'} \in H$ with $\|x^{'}\| = 1$ such that $\|A^{'}x^{'}\| = \|A^{'}\|$. That is, $\|A^{'}A^{'*}x^{'}\| = \|A^{'2}x^{'}\|$. Let
$\eta = \frac{A^{'}x^{'}}{\|A^{'}\|}$, then η is a unit vector such that $\|\eta\| = 1$ this implies that
$\|A^{'*}\eta\| = \|A^{'}\| = \|A^{'*}\|$. Hence, $A^{'*}$ is norm attainable. □

The next result gives norm-attainability conditions for operators via the essential numerical range. An analogy of the same can be found in [65].

Proposition 4.2. *Let* $A^{'} \in B(H)$*,* $\lambda \in W_{ess}(A^{'})$ *and* $\eta > 0$. *Then there exists* $A^{'}_{0} \in B(H)$ *such that* $\|A^{'}\| = \|A^{'}_{0}\|$ *with* $\|A^{'} - A^{'}_{0}\| > \eta$.

Proof. See [65] for the proof. □

Remark 4.3. The set of all norm attainable operators is denoted by $NA(H)$, the set of all norm-attainable self adjoint operators is denoted by $NA^{*}(H)$ and the set of all norm attainable elementary operators is denoted by $E_{NA}[B(H)]$.

At this point, we consider norm-attainability in a general set up. We begin with the following proposition.

Proposition 4.4. *Let D be the unit disc of a complex Hilbert-space H and A* : $H \rightarrow H$ *be compact and self adjoint. Then there exist* $x \in D$ *such that* $\|Ax\| = \|A\|$.

Proof. By the definition of usual norm, we have $\|A\| = \sup_{x \in D} \|Ax\|$. So, there exist a sequence $x_1, x_2, ..., x_n \in D$ such that $\|Ax_n\| = \|A\|$. But A is compact so let $y_0 = \lim_{n\to\infty} Ax_n$ exist in H. Suppose $Y = span\{x_1, x_2\}$, then it is a closed subspace of H. If we pick a subsequence x_{nk} of x_n, then it converges weakly to x and we have done x, $x = \lim_{k\to\infty} x_{nk}$, x and $| x_{nk} , x | \leq \|x_{nk}\| \|x\| = 1$ for all k. Therefore, $\|x\| \leq 1$ but we cannot have $\|x\| < 1$ since then $\|Ax\| = \|A\|\|x\| < \|T\|$ which is a contradiction. Thus, $\|x\| = 1$ that is $x \in D$. Hence, the existence of x is shown and thus completes the proof. □

At this point, we consider q-normality and q-norm-attainability.

Lemma 4.5. *Let $A \in NA(H)$ then A is q-norm attainable if it is q normal.*

Proof. Let $A \in NA(H)$ be q normal that is $A^q A^* = A^* A^q$. Raising A^* to power q and using it to replace A^* we have $A^q (A^*)^q = (A^*)^q A^q$. This shows that A^q is normal. Now $A^q A^* = A^* A^q$ by Fuglede-property. Therefore, A is q normal. However, $A \in NA(H)$ and A^q is normal so it follows that there exist a unit vector $x \in H$ such that $\|A^q x\| = \|A^q\|$, for any $q \in \mathbb{N}$. Hence, A^q is norm attainable. □

Remark 4.6. Every norm attainable operator and every self adjoint op-erator is q-norm attainable and q normal for any $q \in \mathbb{N}$. However, the converse need not be true in general see [66].

Lemma 4.7. *Let $NA_q(H)$ be the set of all q-norm-attainable operators on H. Then $NA_q(H)$is a closed subset of $NA(H)$ which is algebraic if and only if for any $A \in NA(H)$, A is q-normal.*

Proof. Let A be q-normal and pick $\lambda \in \mathbb{K}$. By premultiplying by λ and postmultiplying by q as a power on the normal A we have $(\lambda A)^q (\lambda A)^* =$

$(\lambda A)^*(\lambda A)^q$. This proves the normality of λA. Now if $A \in NA(H)$ then the converse is true if we take limits over a sequence of vectors in H and also by Proposition 4.4. Therefore, A is a q-normal. □

Theorem 4.8. *Let $A \in NA_q(H)$. Then the following conditions are true.*

(i). A^* is q-norm-attainable.

(ii). $V'AV'$ is q-normal, for a unitary operator $V' \in NA_q(H)$.

(iii). A^{-1} is q-norm attainable if it exists.

(iv). $A_0 = A/G$ is q-norm-attainable for some G which is a uniformly invariable subspace of H which reduces to A.

(v). A_0 is uniformly equivalent to A implies A_0 is norm-attainable.

Proof. (i). Since $A \in NA_q(H)$ then from Lemma 4.5 A^q is q-norm at-tainable and so $(A^*)^q$ is norm attainable. Consequently, A^* is q-norm attainable.

(ii). Since V unitary then $VV^* = V^*V = I$, where I is the identity operator. By definition of norm-attainability and Lemma 4.5 we obtain the desired results.

(iii). If A^{-1} exists then since A is q-norm attainable, A^q is q-norm attain-able. Now since A is q-norm-attainable then by Lemma 4.5 A^q is q-norm-attainable. But $(A^q)^{-1} = (A^{-1})^q$ is q-norm-attainable. So A^{-1} is q-norm-attainable.

(iv). Follows from the fact that G invariant under A.

(v). Follows from (iii) since V is unitary.

□

Corollary 4.9. *Let A^q, $A^q_0 \in NA_q(H)$ be commuting operators, then A, $A_0 \in NA_q(H)$.*

Proof. Since A^q, $A^q_0 \in NA_q(H)$ are commuting then A, A_0 are commuting normal operators. By supraposinormality of operators in dense classes we have A, $A_0 \in NA_q(H)$ and hence are norm-attainable. Indeed, $A^qA^q_0 = (AA_0)^q = (A_0A)^q$ which is normal and norm attainable. Hence, A, $A_0 \in NA_q(H)$. □

Remark 4.10. Not all q-norm attainable operators are q normal. Thus, the following example shows that the two commuting q normal operators need not be q normal.

Example 4.11. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Now $A + A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $(A + A_0)^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ are not normal. So $A + A_0$ is not 2-normal. We note that A is self-adjoint.

Lemma 4.12. *The sum of norm-attainable operators is norm attainable.*

Proof. Consider A, $B \in NA(H)$. We need to show that the sum of A and B is norm attainable. For A, B to be norm attainable then there exist a unit vector $x \in H$ such that $\|x\| = 1$, $\|(A + B)x\| = \|Ax + Bx\| = \|A + B\| = \|A\| + \|B\|$. Since $\|Ax + Bx\| \leq \|Ax\| + \|Bx\| \leq \|A\| + \|Bx\| \leq \|A\| + \|B\|$ then for an orthonormal sequence $x_n \in H$ we have $\lim_{n\to\infty}(\|Ax_n + Bx_n\|) = \|Ax + Bx\|$. But since A and B are norm-attainable we have $\|\tilde{A}x + \tilde{B}x\| = \|(\tilde{A} + \tilde{B})x\| = \|\tilde{A} + \tilde{B}\|$ is norm-attainable. □

Theorem 4.13. *A norm-attainable operator perturbed by an identity operators is norm attainable.*

Proof. Let $B \in B(H)$ be norm attainable. Since B is norm attainable then there exist a unit vector $x_0 \in H$, an identity $I \in B(H)$ and for every
$\varepsilon > 0$ we have $\|(BI)x_0\| \leq \|BIx_0\| + \varepsilon \leq \|B\|\|I\|\|x_0\| + \varepsilon$. Since ε is arbitrary then it follows that $\|(BI)x_0\| \leq \|B\|\|I\|\|x_0\| = \|B\|$. Hence,
$\|(BI)x_0\| = \|B\|$. □

At this point, we consider norm attainability for elementary-operators.
We begin with inner derivations.

Lemma 4.14. *Let $\delta_A \in E[B(H)]$, then δ_A is norm attainable if there exists a unit vector $x_0 \in H$, $A \in NA(H)$ and $Ax_0, x_0 \in W_{ess}(A)$.*

Proof. For an operator $A \in NA(H)$ we know that an operator is norm-attainable via essential numerical range from proposition 4.2. Now, we need to show that $\delta_A \in E[B(H)]$ is norm attainable. By the definition of inner derivation, $\delta_A = AY_0 - Y_0A$. Since A is norm attainable then there exist a unit vector $x_0 \in H$ such that $\|x_0\| = 1$, $\|Ax_0\| = \|A\|$. By orthogonality let y_0 satisfy $y_0\ \{Ax_0, x_0\}$ and a contractive Y_0 be defined as a linear transformation $Y_0 : x_0 \to x_0$ with $Ax_0 \to -Ax_0$ as $y_0 \to 0$. Since Y_0 is a bounded linear operator on H, then by norm attainability $\|Y_0x_0\| = \|Y_0\| = 1$ and

$$\|AY_0x_0 - Y_0Ax_0\| = \|Ax_0 - (-Ax_0)\| = 2\|A\|.$$

It follows from Lemma 3.1 in [87] that $\|\delta_A\| = 2\|A\|$. By the inner product $Ax_0, x_0 = 0 \in W_{ess}(A)$, it follows that $\|\delta_A\| = 2\|A\|$. Therefore, $\|AY_0 - Y_0A\| = 2\|A\| = \|\delta_A\|$. Hence, δ_A is norm-attainable. □

Lemma 4.15. *Let $A, A_0 \in B(H)$. If there exist unit vectors y and y_0 on* H *such that A, A_0 are norm-attainable then δ_{A,A_0} is also norm-attainable.*

Proof. Given the operators $A, A_0 \in B(H)$ are norm attainable then we need to show that δ_{A,A_0} is also norm attainable. We define the generalized derivation by $\delta_{A,A_0}(Y) = AY - YA_0$. Since A, A_0 are norm-attainable then there exist unit vectors y and y_0 on H such that $\|y\| = \|y_0\| = 1$, $\|Ay\| = \|A\|$ and $\|A_0y_0\| = \|A_0\|$. If y and Ay are linearly dependent then we have $\|Ay\| = \eta\|A\|y$ where $|\eta| = 1$ and $|Ay, y| = \|A\|$. It follows that $|A_0y_0, y_0| = \|A_0\|$ which implies that $\|A_0y_0\| = \phi\|A_0\|y_0$ and $|\phi| = 1$.
Therefore, $\frac{A_0y_0}{\|A_0\|}, y_0 = \phi = -\frac{Ay}{\|A\|}, y = -\eta$. If Y is defined as $Y : y \to y_0$ and $y_0 \to 0$, $\|Y\| = 1$ then $(AY - YA_0)y_0 = \phi(\|A\| + \|A_0\|)y_0$ which implies $\|AY - YA_0\| = \|(AY - YA_0)y_0\| = \|A\| + \|A_0\| = \|\delta_{A,A_0}\|$. Hence, δ_{A,A_0} is norm-attainable. □

Lemma 4.16. *Every inner derivation is norm attainable if and only if it is self adjoint.*

Proof. Let $\delta_A \in B(H)$ be norm-attainable then we show that $\delta_A = \delta_A^*$. Now since $\delta_A \in B(H)$ is norm attainable then there exist a contraction $Y \in B(H)$ such that $\|\delta_AY\| = \|\delta_A\|$. That is, $\|\delta_A^*\delta_AY\| = \|\delta_A^2Y\|$. Let $\eta \in H$ be defined as $\eta = \frac{\delta_A}{\|\delta_A\|}$ then η is contractive such that $\|\delta_A^*\eta\| = \|\delta_A\| = \|\delta_A^*\|$. Hence, δ_A is self-adjoint. Conversely, let δ_A be self-adjoint. Now since δ_A^* is norm-attainable from the first part, then there exists a contractive $M \in B(H)$ such that $\|\delta_A{}^* M\| = \|\delta_A{}^*\|$, that is $\|\delta_A \delta_A{}^* M\| = \|\delta_A M\|^2$. Let ζ be denoted by $\zeta = \frac{\delta^*_A}{\|\delta_A{}^*\|}$ where $\|\zeta\| = 1$ such that $\|\delta_A \zeta\| = \|\delta_A{}^*\| = \|\delta_A\|$. Hence, δ_A is norm-attainable. □

Lemma 4.17. *Every generalized derivation is norm-attainable if and only if it is implemented by orthogonal projections.*

Proof. Let A, $A_0 \in B(H)$ be orthogonal projections. Indeed, to show that a generalized derivation is implemented by orthogonal projections A and A_0, it is enough to show that it is self adjoint if and only if it is normal as proved in [44]. Let $\delta_{A,A0} : B(H) \to B(H)$ be bounded linear operator on $B(H)$. Then there exist a unique bounded linear operator $\delta_{A,A^*_0} : B(H) \to B(H)$ such that $\delta_{A,A0}\, X, Y = X, \delta_{A,A^*_0}\, Y$, for all $X, Y \in B(H)$. Now,

$$\begin{aligned}\|\delta_{A,A^*_0} Y\| &= \sup_{\|X\|=1} \delta_{A,A0}\, X, Y \\ &\leq \sup_{\|X\|=\|Y\|=1} \|\delta_{A,A0}\| \|X\| \|Y\| \\ &= \|\delta_{A,A0}\|\end{aligned}$$

So, we conclude that δ_{A,A^*_0} is norm attainable. Conversely, let $\delta_{A,A0}$ be norm attainable. We need to show that it is implemented by orthogonal projections. This follows immediately from [44] and this completes the proof. □

4.3 Upper and lower norm estimate for norm-attainable derivations

In this section, we give results on upper and lower norm estimates for norm-attainable derivations. We consider both inner derivations and gen-eralized derivations. We begin with the following proposition.

Proposition 4.18. *Let* $C^{'}, D^{'} \in NA(H)$ *and* $\delta_{C',D'}$ *be bounded then* $\|\delta_{C',D'}\| \leq \|C^{'}\| + \|D^{'}\|$.

Proof. Since $\delta_{C',D'}$ is bounded then for fixed $C^{'}, D^{'} \in NA(H)$ we have $\|\delta_{C',D'}(X)\| \leq \|C^{'}X - XD^{'}\| \leq \|C^{'}X\| + \|XD^{'}\| \leq \|C^{'}\|\|X\| + \|X\|\|D^{'}\|$. Let X be of norm 1 and take supremum over $X \in NA(H)$ then $\|\delta_{C',D'}\| \leq \|C^{'}\| + \|D^{'}\|$. □

Remark 4.19. If $C^{'} = D^{'}$ then $\|\delta_{C'}\| \leq 2\|C^{'}\|$.

Next, we consider upper bounds in the unit ball of $NA(H)$ denoted by $[NA(H)]_0$.

Lemma 4.20. *Let* $[NA(H)]_0$ *be the unit ball of* $NA(H)$ *and* S *be a fixed element of* $NA(H)$. *Let* $X \in [NA(H)]_0$ *then* $\|\delta_S|_{[NA(H)]_0}\| \leq 2d(S)$.

Proof. Since $X \in [NA(H)]_0$ has norm 1 then we have $\|\delta_S|_{[NA(H)]_0}(X)\| = \|SX-XS\|_{[NA(H)]_0} = \|(S-\lambda)X-X(S-\lambda)\|_{[NA(H)]_0} \leq \|S-\lambda\|\|X\|_{[NA(H)]_0} + \|X\|\|S-\lambda\|_{[NA(H)]_0}$. Take the supremum over $X \in [NA(H)]_0$, we obtain $\|\delta_S|_{[NA(H)]_0}\| \leq 2\|S-\lambda\|$ and considering the infimum over $\lambda \in \mathbb{C}$ we obtain $\|\delta_S|_{[NA(H)]_0}\| \leq 2 \inf_{\lambda \in \mathbb{C}} \|S-\lambda\| = 2d(S)$. □

Remark 4.21. The restriction of $\delta_A|_{[NA(H)]_0}$ i.e δ_A to $[NA(H)]_0$ is a bounded linear operator.

Next we give an extension of Lemma 4.20 to a generalized derivation in the following theorem.

Theorem 4.22. *Let S, S_0 be fixed elements of $NA(H)$ then*

$$\|\delta_{S,S_0}|_{[NA(H)]_0}\| \leq \|\delta_{S,S_0}\|.$$

Proof. Since $X \in [NA(H)]_0$ has norm 1 then we have $\|\delta_{S,S_0}|_{[NA(H)]_0}(X)\| = \|SX - XS_0\|$. Following proof of lemma 4.20 anologously we have

$$\|\delta_{S,S_0}|_{[NA(H)]_0}(X)\| \leq \|S - \lambda\|\|X\|_{[NA(H)]_0} + \|X\|\|S_0 - \lambda\|_{[NA(H)]_0}.$$

Taking the supremum over $X \in [NA(H)]_0$ we obtain

$$\|\delta_{S,S_0}|_{[NA(H)]_0}\| \leq \inf_{\lambda\in\mathbb{C}}(\|S - \lambda\| + \|S_0 - \lambda\|) = \|\delta_{S,S_0}\|. \qquad \square$$

Corollary 4.23. *Every generalized derivation δ_{S,S_0} is norm-bounded.*

Proof. This follows immediately from [87] and from Theorem 4.22. This completes the proof. □

Proposition 4.24. *Let S, S_0 be fixed elements of $NA(H)$ then*

$$\|\delta_{S,S_0}|_{[NA(H)]_0}\| \geq \|S\| + \|S_0\|.$$

Proof. Let η, ξ and x be unit vectors in H and ϕ, φ be positive linear functionals such that $\phi \otimes \eta : H \to \mathbb{C}$ and $\varphi \otimes \xi : H \to \mathbb{C}$ be of rank 1 defined as $(\phi\otimes\eta)x = \phi(x)\eta$ and $(\varphi\otimes\xi)x = \varphi(x)\xi$, $\forall x \in H$, $\|x\| = 1$. Now we have that $\|(\phi \otimes \eta)x\| = \sup\{ (\phi \otimes \eta)x\|, \|x\| = 1\} = |\phi(x)| = |\phi|$. Similarly, we have $\|(\varphi\otimes\xi)x\| = \|\varphi\|$. Letting $S = \phi\otimes\eta$ and $S_0 = \varphi\otimes\xi$ then $\|S\| = \|\phi\|$ and $\|S_0\| = \|\varphi\|$. Now from Corollary 4.23 we have that every general-ized derivation is norm-bounded this implies that $\|\delta_{S,S_0}|_{[NA(H)]_0}(X)\| \geq \|\delta_{S,S_0}(X)\|$ where $X \in [NA(H)]_0$. Therefore, $\|\delta_{S,S_0}|_{[NA(H)]_0}\|^2 \geq \|SX - XS_0\|^2$ implying that $\|\delta_{S,S_0}|_{[NA(H)]_0}\|^2 \geq [\|S\| + \|S_0\|]^2$. Taking positive square root on both sides we obtain $\|\delta_{S,S_0}|_{[NA(H)]_0}\| = \|\delta_{S,S_0}\| \geq \|S\| + \|S_0\|$. □

Remark 4.25. If $S = S_0$ then $\|\delta_{S,S_0}\| = \|\delta_S\| \geq 2\|S\|$.

Remark 4.26. From Theorem 4.22 and Proposition ?? it is easy to see that $\|\delta_{S,S_0}\| = \|S\| + \|S_0\|$ and hence $\|\delta_S\| = 2\|S\|$.

Theorem 4.27. *Let* $S, S_0 \in NA(H)$ *and* $\alpha_1 \in W_0(S)$ *and* $\alpha_2 \in W_0(S_0)$. *Then* $\|\delta_{S,S_0}\| \geq (\|S\|^2 - |\alpha_1|^2)^{1/2} + (\|S_0\|^2 - |\alpha_2|^2)^{1/2}$.

Proof. By definition of $W_0(S)$ we have $x_n \in H$ such that $\|Sx_n\| = \|S\|$ and $Sx_n, x_n \to \alpha_1$ for $\alpha_1 \in W_0(S)$. This argument follows for $W_0(S_0)$ and $\alpha_2 \in W_0(S_0)$. Let $Sx_n = \delta_n x_n + \beta_n y_n$ so $S_0 x_n = \sigma_n x_n + \lambda_n y_n$ where $x_n, y_n = 0$, $\|y_n\| = 1$. Take $U_n x_n = x_n$ and $U_n y_n = -y_n$ for $U_n = 0$ in $\{x_n, y_n\}$. Then $\|SU_n x_n - U_n S_0 x_n\| = \|\delta_n + \beta_n\| \leq |\delta_n| + |\beta_n|$. But $|\delta_n| + |\beta_n| \geq (\|S\|^2 - |\delta_n|^2)^{1/2} - \xi_n + (\|S_0\|^2 - |\beta_n|^2)^{1/2} - \xi_n)$. Since ξ_n is arbitrary and letting $n \to \infty$, so it follows that $\|\delta_{S,S_0}\| \geq \|(SU_n - U_n S_0)x_n\| = |\delta_n| + |\beta_n| = (\|S\|^2 - |\alpha_1|^2)^{1/2} + (\|S_0\|^2 - |\alpha_2|^2)^{1/2}$. □

Corollary 4.28. *Let* $x_n, y_n = 0$ *then* $0 \in W_0(S)$ *and if* $0 \in W_0(S_0)$ *then* $\|\delta_{S,S_0}\| \geq \|S\| + \|S_0\|$.

Proof. Follows immediately from definition of $W_0(S)$ and the Theorem 4.27.

□

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this chapter, we give the conclusion and recommendations based on the objectives of the study and the results obtained on norm-attainability conditions, upper norm estimates and lower norm estimates for norm-attainable derivations in Banach algebras.

5.2 Conclusion

We give the conclusion regarding the problem stated on Section 1.3 of our work by highlighting the results obtained in our study .

In objective one, we established norm-attainability conditions and concluded that $\|AY_0x_0 - Y_0Ax_0\| = \|Ax_0 - (-Ax_0)\| = 2\|A\| = \|\delta_A\|$.

In objective two, we determined the upper norm estimates for norm-attainable derivations and showed that $\|\delta_{A,B}\| \leq \|A\| + \|B\|$ and for the lower norm estimates for norm-attainable derivations we showed that $\|\delta_{S,S_0}\| \geq \|S\| + \|S_0\|$.

Therefore, we have given results on norm-attainability conditions for derivations, the upper and lower norm estimates for norm-attainable derivations in Banach algebras.

5.3 Recommendations

In objective one, we have established results on norm-attainability conditions for derivations in Banach algebras. We recommend that further study can be done to establish norm-attainability conditions for deriva-tions when they are implemented by transaloid operators normaloid op-erators.

In objective two, we have determined upper and lower norm estimates for norm-attainable derivations in Banach algebras. We recommend that further studies can be done to determine upper and lower norm estimates for norm-attainable derivations when they are implemented by transaloid operators and normaloid operators.

References

[1] Archbold R. J., On the Norm of an inner derivation of a C^*-Algebra. *Math. Proc. Camb. Phil. Soc.*, Vol. 84, No.2, (1978), 273-
291.

[2] Abolfazl N. M., On the norm of Jordan $*$-derivations. *Khaayyam J. Math.*, Vol. 6, No. 1, (2020), 104-107.

[3] Abramovich A. Y., Aliprantis C. D., Burkinshaw O., The Daugavet equation in uniformly convex Banach spaces. *J. Funct. Anal.*, 97, (1991), 215-230.

[4] Alexander G., Nonparametric estimation of transfer functions: rates of convergence and adaptation. *IEEE Transactions on Infor-mation Theory*, Vol. 44, No. 2, (1998), 644-658.

[5] Anderson J. H., On normal derivations. *Proc. Amer. Math. Soc.*, 38 (1973), 135-140.

[6] Baxter B. J. C., Norm estimates for inverses of Toeplitz distance Matrices. *J. of Approximation Theory*, 79, (1994), 222-242.

[7] Barraa M., Boumazgour M., Inner derivations and norm equal-ity. *Pro. Amer. Math. Soc.*, Vol. 130, No. 2, (2001), 471-476.

[8] Ber A. F., Sukochev F. A., Commutator estimates in W^*-factors. *Trans. Amer. Math. Soc.*, Vol. 364, No. 10, (2012), 5571-5587.

[9] Bonyo J. O., Agure J. O., Norms of derivations implemented by S-universal operators. *Int. Journal of Math. Analysis,* Vol. 5, No. 5, (2011), 215-222.

[10] Bonyo J. O., Agure J. O., Norm of a derivation and hyponormal operators. *Int. Journal of Math. Analysis*, Vol. 4, No. 14, (2010), 687-693.

[11] Bonyo J. O., Agure J. O., Norms of Inner derivations on norm ideals. *Int. Journal of Math. Analysis*, Vol. 4, No. 14, (2010), 695-701.

[12] Blanco A., Boumazour M., Ransford T. J., On the norms of the elementary operators. *J. London Math. Soc.*, 70, (2004), 479-498.

[13] Bresar M., Zalar B., On the structure of Jordan $*$-derivations. emphColloq. Math., 63, No. 2, (1992), 163-171.

[14] Briggs J. M., Approximation with norms defined by derivations. *J. Austral. Math. Soc. 20(series A),* (1975), 18-24.

[15] Cabrera M., Rodriguez A., Nondegenerately ultraprint Jordan Banach algebras. *Proc. London Math. Soc.*, 69, (1994), 576-604.

[16] Charles A. A., Steve W., Compact and weakly compact deriva-tions of $C*$-algebras. *Pacific Journal of Math.*, vol. 85, No. 2, (1979).

[17] Chi-Kwong L., Lecture notes on numerical range. (2005), 1-25.

[18] Clifford G., Dynamics of generalized derivations and elementary operators. *arXiv:1605.07409v2 [math.FA]* (2017), 1-17.

[19] Cristina B., Camil M., Mixed-norm estimates via the Helicoidal method. *arXiv:2007.01080v1 [math.CA]*, (2020), 1-48.

[20] Danko R. J., Norm Inequalities for self-adjoint derivations. *Journal of Functional Analysis*, Vol. 145, No. 4, (1997), 24-34.

[21] Douglas W. B. S., The inner derivations and the primitive ideal space of a C*-algebra. *J. Operator Theory*, 29 (1993), 307-321.

[22] Dutta T. K., Nath H. K., Kalita R. C., α-derivations and their norm projective tensor products of Γ-Banach algebras. *Int. J. Math. Sci.,* Vol. 21, No. 2, (1998), 359-368.

[23] Erik C., Extensions of derivations II. *Math. Scand.*, 50, (1982), 111-122.

[24] Elena C., Lorenza D., Ivan S. T., Norm estimates for τ-pseudodifferential operators in wiener amalgam and modulation. *ariv:1803.07865v1[math.FA]*, 21, (2018).

[25] Fong C. K., Norm estimates related to self-commutators. *Linear algebra and its applications*, 74, (1986), 151-156.

[26] Florin P. B., Alexandra Z., Norm estimates of almost Mathieu operators. *Journal of Functional Analysis,* 220 (2005) 76-96.

[27] Gajendragadkar P., Norm of a derivation on Von Neumann alge-bra. *Transactions of the American Mathematical Society*, Vol. 170, (1972).

[28] Gyan P. T., Maximal numerical range of composition operators on l^2. *Gatinah*, Vol. 70 (1), (2020), 65-74.

[29] Gil M., Norm estimates for functions of two commuting matrices. *Electronic Journal of Linear Algebra*, Vol. 13, No. 8, (2005), 122-130.

[30] Gil' M. I., Estimates for norm of matrix-valued functions. *Linear and Multilinear Algebra*, 35, (1993), 65-73.

[31] Gustafson K. E., Rao D. K. M., Numerical range: the field of values of linear operators and matrices. *Springer-Verlag, New York*, (1997).

[32] Hoger G., Jordan derivation on trivial extensions.*Bull. Iran. Math. Soc.*, Vol. 39, No. 4, (2013), 635-645.

[33] Hong-Ke D., Yue-qing W., Gui-Bao G., Norms of elementary operators. *Proc. Amer. Math. Soc.*, Vol. 136, No. 4, (2008), 1337-1348.

[34] Jian-Feng Z., David K., Norm estimates of the cauchy transform and related operators. *arXiv:2008.02524v1[math.CV]*, (2020), 1-20.

[35] Joel H. S., Notes on numerical range. *Michigan State Univer-sity,East Lansing*, (2003), 1-15.

[36] Johnson B., Characterization and norms of derivations on Von Neu-mann algebras. Vol. 1, No.1, (1979), 228-236.

[37] Johnson B. E., Norms of derivations on $L(X)$. *Pacific Journal of Math*, Vol. 38, No. 2, (1971), 465-469.

[38] Kyle J., Numerical ranges of derivations. *proc. of the Edinburgh Math. Soc.*, Vol. 21, (1978), 33-39.

[39] Kyle J., Norms of derivations. *J. London Math. Soc.*, Vol. 16, No. 2, (1977), 297-312.

[40] Kittaneh F., Normal derivations in norm ideals. *Pro. Amer. Math. Soc.*, Vol. 123, No. 6, (1995), 1779-1785.

[41] Kinyanjui J. N., Okelo N. B., Ongati O., Musundi S. W., Norm estimates for norm-attainable elementary operators. *Int. J. Math. Anal,* Vol. 12, No. 3, (2018), 137-144.

[42] Kinyanjui J. N., Okelo B., Ongati O., Musundi S., Charac-terization of norm-attainable operators. *Int. J. Math. Archive*, Vol. 9, No.7, (2018), 124-129.

[43] Kingangi D. N, Agure J. O., Nyamwala F. O., On the norm of elementary operator. *Advances in Pure Math.*, Vol. 4, (2014), 309-316.

[44] Kreyszig E., Introduction Functional Analysis with Applications. *Book.Canada publications*, (1978), 1-703.

[45] Landsman N. P., C*-algebras and quantum mechanics. Lecture notes. (1998).

[46] Lumer G. Complex methods and the estimation of operator norms and spectra from real numerical ranges. *J. Functional Anal.*, Vol. 10, (1972) 482-495.

[47] Matej B., On distance of the composition of two derivations to the generalized derivations. *Glasgow Math. J.*, 33,(1991), 89-93.

[48] Mathieu M., More properties of the product of two derivations of a C^*-algebra. *Bull. Austral. Math. Soc.,* Vol. 42, (1990), 115-120.

[49] Mathieu M., Elementary operators on Calkin algebras. *Irish Math. Soc. Bulletin*, Vol. 46, (2001), 33-44.

[50] Man-Duen C., Chi-Kwong L., The ultimate estimate of the up-per norm bound for the summation of operators. *J. of Functional Anal.* 232 (2006) 455-476.

[51] Megginson R. E., An introduction to Banach space theory. *Springer-Verlag, New York*, (1998).

[52] Milos A., Dragoljub K., Elementary operators on Banach algebras and fourier transform. *Math. Subject Class.* Vol. 42, No. 10, (2000), 1-23.

[53] Mecheri S., The Gateaux derivative orthogonality in C_∞. Lecture notes. (1991).

[54] Nyamwala F. O., Agure J. O., Norms of elementary operators in Banach algebras. *Int. J. Math. Analysis*, Vol. 2, (9), (2008), 411-424.

[55] Nyamwala F. O., Norms of symmetrised two-sided multiplication operators. *Int. J. Math. Anal.*, Vol. 3, No.35, (2009), 1735-1744.

[56] Odero A. B., Agure J. O., Nyamwala F. O., On the norm of a generalized derivation. *Pure Math. Sci.*, Vol. 8, No. 1, (2019), 11-16.

[57] Odero A. B., Agure J. O., Rao G. K., Norms of tensor products elementary operators. *Int. J. of Multidisciplinary Sci. and Engineer.*, Vol. 6, No. 10, (2015), 29-32.

[58] Okelo N. B., Okongo M. O., Nyakiti S. A., On projective tensor norm and norm-attainable α-derivations. *Int. J. Contemp. Math. Sciences*, Vol. 5, No. 40, (2010), 1969-1975.

[59] Okelo N. B., Mogotu P. O., On norm inequalities and orthog-onality of commutators of derivations. *ArXiv:1903.10358V1 Math. FA*, No. 25, 2019.

[60] Okelo N. B., On orthogonality of elementary operators in Norm-attainable classes. *Taiwanese J. of Math.*, Vol. 24, No. 1, (2020), 119-130.

[61] Okelo N. B., Agure J. O., Ambogo D. O., Norms of elemen-tary operators and characterization of norm-attainable operators. *Int. Journal of Math. Analysis*, Vol. 4, No. 24, (2010), 1197-1204.

[62] Okelo N. B., Ongati O., Obogi R. K., Projective norms and convergence of norm-attainable operators. *Journal of Global Research in Mathematical Archives*, Vol. 2, No. 5, (2014), 10-17.

[63] Okelo N. B., Various notions of norm-attainability in normed spaces. *ArXiv:2004.05496V1 [Math. FA]*, 11 Apr (2020), 1-13.

[64] Okelo N. B., Norm-attainability and range-kernel orthogonality of elementary operators. *Communications in Advanced Mathematical sciences*, Vol.1, No. 2, (2018), 91-98.

[65] Okelo N. B., The norm-attainability of some elementary operators. *Appied Math. E-Notes*, Vol. 13, (2013), 1-7.

[66] Okelo N. B., Agure J. O., Oleche P. O., Certain conditions for norm-attainability of elementary operators and derivations. *Int. J. Math. and Soft Computing*, Vol. 3, No. 1, (2013), 53-59.

[67] Okelo N. B., Aminer T. J. O., Norm inequalities of norm-attainable operators and their orthogonality extensions. (2020), 1-9.

[68] Okelo N. B., Ambogo D. A., Nyakiti S. A., On the constants $C(\Omega)$ and $C_S(\Omega)$ of a C^*-algebra and norms of derivations. *Int. Math. Forum*, 5, No. 53, (2010), 2647-2653.

[69] Okelo N. B., Agure J. O. A two-sided multiplication operator norm. *Gen. Math. notes*, Vol. 2, No.1, (2011), 18-23.

[70] Okelo N. B., Fixed points approximation for nonexpansive opera-tors in Hilbert spaces. *Int. J. Open Problems Compt.*, Vol. 14, No. 1, (2021), 1-5.

[71] Okelo N. B., Characterization of absolutely norm attaining com-pact hyponormal operators. *Proc. Int. Math. Sci.*, Vol. II, Issue 2, (2020), 96-102.

[72] Okelo N. B., Norms and norm-attainablity of normal operators and their applications. *JKUAT Sci. Conf.*, (2015), 83-86.

[73] Ola B., Akataka K. Derek W. R., Approximately inner deriva-tions. *aXiv:math./0703013v1 [math. OA]*, (2007), 1-17.

[74] Oyake M. O., Okelo N. B., Ongati O., Characterization of inner derivations induced by norm-attainable operators. *Int. J. of Modern Science and Technology*, Vol. 3, No. 1, (2018), 6-9.

[75] Pablo G., Jussi L., Mikael L., Essential norm estimates for com-position operators on *BM OA. J. of Functional Anal.*, 265, (2013), 629-643.

[76] Pinchuck A., *Functional analysis notes* 2011.

[77] Rajendra B., Kalyan B. S., Derivations, derivatives and chain rules. *Linear Algebra and its Applications*, 302-303, (1999), 231-244.

[78] Richard M. T., Norms and CB norms of Jordan elementary oper-ators. *Bull. Sci. Math.*, 127 (2003), 597-609.

[79] Richard M. T., Computing the norms of elementary operators. *Illinois J. Math.*, Vol. 47, No. 4, (2003), 1207-1226.

[80] Sayed K. E., Madjid M., Hamid R. E. V., Product of C^*-algebras. *Int. J. Nonlinear Anal. Appl.*, Vol. 7, No. 2, (2016), 109-
114.

[81] Seddik A., On the numerical range and norm of elementary op-erators. *Linear and Multilinear Algebra*, Vol. 52, No. (3-4), (2004), 293-302.

[82] Seddik A., On the injective norm and characterization of some subclasses of normal operators by inequaities or equalities. *J. Math. Anal. Appl.*, 351 (2009), 277-284.

[83] Sen-Yen S., On numerical ranges of generalized derivations and related properties. *J. Austal. Math. Soc. (series A)*, 36, (1984), 134-142.

[84] Siva R.,Richard F. B., Edwin A. P., H¨*o*lder norm estimates for elliptic operators on finite and infinite-dimensional spaces. *Transac-tion Amer. Math. Soc.*, Vol.357, No. 12, (2005), 5001-5029.

[85] Shlomo R., Distance estimates for Von Neumann algebras. *Pro. Amer. Math. Soc.*, Vol. 86, No. 2, (1982), 248-252.

[86] Shinji Y., Norm estimates for funtion starlike or convex of order alpha. *Hokkaido Math. J.*, Vol. 28, (1999), 217-230.

[87] Stampfli J. G., The norm of a derivation. *Pac. J. Math.*, Vol. 33, No. 3, (1970), 737-747.

[88] Stampfli J. G., On selfadjoint derivation ranges. *Pacific J. of Math.* Vol. 82, No. 1, (1979), 257-277.

[89] Stacho L. L., Zalar B., On the norm of Jordan elementary oper-ators in standard operator algebra. *Publ. Math. Debrecen*, Vol. 49, No. (1-2), (1996), 127-134.

[90] Stephen M. B., Estimates for operator norms on weighted spaces and reverse Jensen inequalities. *Transactions Amer. Math. soc.*, Vol. 340, No. 1, (1993), 253-272.

[91] Triet P., Jianfeng Z., Some norm estimates for semimartangales. *Electron J. Probab.* 18, No. 109, (2013), 1-25.

[92] Volker R., Automatic continuity of derivations and epimorphisms. *Pacific J. of Math.*, Vol. 147, No. 2, (1991), 365-374.

[93] Wafula A.M., Okelo N.B., Ongati O., Norms of normally rep-resented elementary operators. *Int. J. of Modern Science and Tech-nology*, Vol. 3, No. 1, (2018), 10-16.

[94] Wickstead A. W., Norms of basic elementary operators on algebras of regular operators.*Proc. Amer. Math. Soc.*, Vol. 143, No.12, (2015), 5275-5280.

[95] Yong C. K., Toshiyuki S., Norm estimates of the Pre-schwarzian derivatives for certain classes of univalent functions. *Proc. of the Edinburgh Math. Soc.*, 49, (2006), 131-143.

APPENDIX 1: INTRODUCTION LETTER.

KISII UNIVERSITY

Telephone: +25420 2352059
Facsimile: +254020 2491131
Email: research@kisiiuniversity.ac.ke

P O BOX 408 – 40200
KISII
www.kisiiuniversity.ac.ke

OFFICE OF THE REGISTRAR RESEARCH AND EXTENSION

REF: KSU/R&E/ 03/5/526 **DATES:** 29th March , 2021

The Head, Research Coordination
National Council for Science, Technology and Innovation (NACOSTI)
Utalii House, 8th Floor, Uhuru Highway
P. O. Box 30623– 00100
NAIROBI - KENYA.

Dear Sir/Madam,

RE: JANES NYABONYI ZACHARY MPS17 /00004/19

The above mentioned is a student of Kisii University currently pursuing a Degree of Master of Science in Pure Mathematics . The topic of her research is, ***"Norm estimates for norm-attainable derivations in Banach algebras"***.

We are kindly requesting for assistance in acquiring a research permit to enable her carry out the research.

KISII UNIVERSITY
29 MAR 2021
REGISTRAR RESEARCH & EXTENSION

Thank you.

for Prof. Anakalo Shitandi, PhD
Registrar, Research and Extension

Cc: DVC (ASA)
Registrar (ASA)
Director SPGS

APPENDIX 2: Permit From NACOSTI

REPUBLIC OF KENYA

NATIONAL COMMISSION FOR SCIENCE,TECHNOLOGY & INNOVATION

Ref No: **485053**

Date of Issue: **26/November/2021**

RESEARCH LICENSE

This is to Certify that Ms.. JANES NYABONYI ZACHARY of Kisii University, has been licensed to conduct research in Kisii on the topic: NORM ESTIMATES FOR NORM-ATTAINABLE DERIVATIONS IN BANACH ALGEBRAS for the period ending : 26/November/2022.

License No: **NACOSTI/P/21/14620**

485053

Applicant Identification Number

Director General
NATIONAL COMMISSION FOR SCIENCE,TECHNOLOGY & INNOVATION

Verification QR Code

NOTE: This is a computer generated License. To verify the authenticity of this document, Scan the QR Code using QR scanner application.

APPENDIX 3: PLAGIARISM REPORT

NORM ESTIMATES FOR NORM-ATTAINABLE DERIVATIONS IN BANACH ALGEBRAS

ORIGINALITY REPORT

20%	17%	13%	3%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

MATCHED SOURCE

1	www.coursehero.com Internet Source	3%

3%

★ www.coursehero.com
Internet Source

Exclude quotes	On	Exclude matches	Off
Exclude bibliography	On		