ON NUMERICAL RANGES AND SPECTRA OF POSINORMAL OPERATORS

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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Dedication

To my beloved wife, Becky Kerubo Okeri and children Stacy, Rodney and Sheryl.

Abstract

Hilbert space operators have been studied by many mathematicians. These operators are of great importance since they are useful in formulation of principles of mathematical analysis and quantum mechanics. The operators include normal operators, posinormal operators, hyponormal operators, normaloid operators among others. Certain properties of posinormal operators have been characterized like continuity and linearity but numerical ranges and spectra of posinormal operators have not been considered. Also the relationship between the numerical range and spectrum has not been determined for posinormal operators. The objectives of this study have been: to investigate numerical ranges of posinormal operators, to investigate the spectra of posinormal operators and to establish the relationship between the numerical range and spectrum of a posinormal operator. The methodology involved use of known inequalities like Cauchy-Schwartz inequality and the polarization identity to determine the numerical range and spectrum of posinormal operators and our technical approach involved use of tensor products. We have shown that the numerical range of a posinormal operator A is nonempty, contains zero and is an ellipse whose foci are the eigenvalues of A. We have also proved that the spectrum of a bounded posinormal operator A acting on a complex Hilbert space H satisfies Xia's property; and doubly commuting n-tuples of posinormal operators are jointly normaloid. The results obtained are applicable in classification of Hilbert space operators and shall be applied in other fields like quantum information theory to optimize minimal output entropy of quantum channel; to detect entanglement using positive maps; and for local distinguishability of unitary operators.

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Chapter 1

INTRODUCTION

1.1 Mathematical background

The study of numerical range and numerical radius has a long and distinguished history[3],[56],[41],[58]. The notion of numerical range was introduced by Toeplitz [55]. It has been investigated extensively because it is a useful tool for studying matrices and operators. It relates to other fields like operator theory, dilation theory, iterations, Krein space operators, numerical analysis, matrix norms, inequalities, Banach algebras, C^* algebras, purturbation theory, systems theory, quantum physics just to mention a few [37],[14]. In regard of the many directions of active research in numerical range and numerical radius, there has been much interest in characterizing both real and complex operators. Meng [39] worked on a condition that a normal operator must have a closed numerical range. Meng showed that, if an operator is normal and its numerical range is closed, then the extreme points of the numerical range are eigenvalues. Meng investigated the numerical range of normal operators on a Hilbert space but not numerical range of posinormal operators. Toeplitz and Hausdorff showed that the numerical range of every bounded linear operator is convex [25],[12]. This result is an important tool in the study of numerical ranges of operators and it holds for all operators on Hilbert spaces. Shapiro [47] proved the Toeplitz-Hausdorff theorem, the Folk theorem and determined the numerical range of a 2×2 matrix. It was proved that the spectrum of an operator is contained in the closure of its numerical range. Hildebrandt had obtained a result that for a bounded linear operator T on a Hilbert space, a convex hull of $\sigma(T)$ can be obtained by intersecting the closures of the numerical ranges of all the operators similar to T [47]. In [47] Shapiro gave a short and complete proof to Hildebrandt's result. Shapiro investigated the numerical ranges of a two dimensional Hilbert space especially 2×2 matrices but not numerical ranges of posinormal operators on an infinite dimensional Hilbert space. In [5] Bebiano investigated numerical ranges associated with operators on an indefinite inner product space . Boundary generating curves, corners, shapes and computer generations of these sets were sudied. Bebaino generalized the Murnaghan- Kippenhahn theorem for classical numerical range. Bebiano [6]investigated the geometrical properties of the classical numerical range and remarked that the classical numerical range of an operator T if a singleton if and only if T is a scaler matrix, moreover, $W(T) \subseteq R$ if and only if T is Hermitian. On computer generations he studied algorithms and computer programs for generating the classical numerical range and its generalizations. He presented an algorithm providing the boundary generating curve of the J- numerical range. Matlab programs to plot this curve and draw an approximation for $W_j(T)$ were developed. Stampfli [52],[53] investigated normality and the numerical range of an operator. The results of Donoghue [15], Hildebrandt, Meng and Putnam were generalized by Stampfli. Stampfli studied the convexity of a curve in the complex plane and remarked that if, at every point, the curve and the origin lie on the same side of the support line, the curve is said to be convex with respect to the origin. Embry [18] studied the numerical range of an operator on a complex Hilbert space. She worked on the property of linearity of the numerical range of operators on a complex Hilbert space, extreme points of the numerical range, boundary points and the convexity property of the numerical range of operators. Embry gave a generalization of Stampfli's result [53] that if an operator T is hyponormal and z is an extreme point (that is, z is not in the interior of any line segment with endpoints in the numerical range of T), then z is an eigenvalue of the operator T. A proof of the convexity of the numerical range of bounded operators popularly known as the Toeplitz- Hausdorff theorem is also given. Embry investigated linearity property and extreme points of the numerical range of complex Hilbert space operators like normal and hyponormal operators but not the numerical range of posinormal operators. Mecheri [37] studied the numerical range of linear operators and gave a necessary and sufficient condition for an operator to be convexoid. The numerical ranges of linear operators were used to prove that convexoid operators on a complex Hilbert space are normaloid and Mecheri gave an example showing that that the converse of this does not necessarily hold. In [4] Barraa studied the essential numerical range of elementary operators particularly the essential numerical range of the restriction of an elementary operator to the class of Hilbert- Schmidt. Johnson [29] characterized normal matrices in terms of the numerical range. A characterization of matrices for which the numerical range coincides with the convex hull of the spectrum is also given. Johnson made a key remark that the eigenvectors corresponding to any eigenvalue occurring on the boundary of the numerical range must be orthogonal to the eigenvectors corresponding to all other eigenvalues. Guediri [20] investigated qualitative properties of the numerical range of dual Toeplitz operators. Various classes of dual Toeplitz operators were considered such as normal and quasinormal dual Toeplitz operators. Guediri gave a complete description of the numerical ranges of dual Toeplitz operators. The main qualitative properties of the numerical ranges of dual Toeplitz operators were established. Guediri further shed some light on the analog of Halmos' fifth problem on the classification of subnormal Toeplitz operators. However, Guediri characterized normality and quasinormality and numerical ranges of some classes of dual Toeplitz operators but not the numerical ranges of posinormal operators. In [13] Ching- Kwong gave a characterization to the points in the numerical range of a Hilbert space operator that lie on the boundary. It was shown that the collection of such boundary points together with the interior of the the convex hull of the spectrum of the Hibert space operator will then be the numerical range of that operator. Moreover, Ching- Kwong showed that such boundary points reveal a lot of information about the normal operator. For instance, such a boundary point always associates with an invariant (reducing) subspace of the normal operator. He further proved that a normal operator acting on a separable Hilbert space cannot have a closed strictly convex set as its numerical range. Further extension of the results obtained to the joint numerical range of commuting operators was discussed. Ching- Kwong investigated and gave a characterization of boundary points in the numerical range of a Hilbert space operator but not the numerical range of a posinormal operator. Recently, Hwa-Long and Pei had excursions in numerical ranges [24]. Anderson's condition for the numerical range of a finite matrix to equal a circular disc was studied. They surveyed Holbrook's conjecture on the numerical radius inequality concerning the product of two community operators. Lastly Hwa-Long and Pei investigated Williams and Crimmins's structure theorem on an operator when its numerical radius equals half of its norm. The excursion concentrated on investigations concerning the classical numerical ranges of operators and finite matrices but not numerical ranges of posinormal operators. Numerical ranges for several Hilbert space operators have been established but not for posinormal operators. In this study we shall investigate the numerical ranges of posinormal operators.

Spectral theory of linear operators on Hilbert spaces is a pillar in several developments in mathematics, physics and quantum mechanics. Its concepts like the spectrum of a linear operator, eigenvalues and vectors, spectral radius, spectral integrals among others have useful applications in quantum mechanics, a reason why there is a lot of current research on these concepts and their generalizations. In [3],[33] spectral theory is described as a rich and important theory as it relates perfectly with other areas including measure and integration theory and theory of analytic functions. Spectral theory of linear operators on a Hilbert space was founded by Hilbert [59]. Weyl advanced the spectral theory for singular second order differential equations [17][26]. In [44] Rhaly introduced a posinormal operator, showed a characterization of posinormality and gave some spectral properties of posinormal operators. The relationship between a hyponormal operator and a posinormal operator was established. Rhaly further introduced a superclass of the posinormal operators and determined sufficient conditions for this superclass to be posinormal and hyponormal [45]. However, Rhaly in his work has not characterized numerical ranges and spectra of posinormal operators, and the relationship between the numerical range and the spectrum of posinormal operators has not been established. In [27] Itoh gave a different characterization of posinormal operators complementing Rhaly's. Lee [34] studied powers of p-Posinormal operators and showed that if T is p- Posinormal then T^n is also p-Posinormal for all positive integer n. Duggal and Kubrusly [17] investigated Weyl's theorems for posinormal operators. They proved that posinormal operators, both totally posinormal operators and conditionary totally posinormal operators, satisfy the Weyl's theorem. Duggal and Kubrusly gave a remark that the restriction of a posinormal operator to an invariant subspace is again posinormal[7]. In [38] Mecheri studied the generalized Weyl's theorem for posinormal operators and proved that the generalized Weyl's theorem holds for $f(T)$ if T is conditionally totally posinormal or totally posinormal, where f is a function analytic in an open neighbourhood of $\sigma(T)$. A definition of posinormality is given equivalent to the one given by Rhaly, that is, an operator T is said to be posinormal if there exists a co-isometry V^* and a positive bounded Hilbert space operator P such that $T = T^*PV^*$. It was noted that the large class of posinormal operators contains other classes such as; the classes consisting of hyponormal operators, M- hyponormal operators and dominant operators. Rhaly [46] introduced a superclass of the posinormal operators which he referred to as Supraposinormal operators. He determined sufficient conditions for a supraposinormal operator to be posinormal and hyponormal. Rhaly [46] gave a brief proof of a well known result, the hyponormality of C_k (the generalized cesàro operator of order one) for $k \geq 1$. Rhaly established a connection between this superclass and some recently published sufficient conditions for a lower triangular factorable matrix to be a hypornomal bounded linear operator on l^2. It was shown that all injective unilateral weighted shifts are supraposinormal. Kostov and Todorov [30] introduced a class of operators called polynormially posinormal operators, which is naturally extending the classes of hyponormal and posinormal operators. They constructed a generating family of eigendistributions, unitary invariants and developed a functional model for this class. It was noted that by extending the class of hyponormal operators to the class of operators possessing the property $\mathrm{Im}T \subset \mathrm{Im}T^*$, posinormal operators are obtained. The class of polynormially posinormal operators includes all finite- dimensional and nilpotent operators and all posinormal operators making it larger than the class of M- hyponormal operators. In [49] spectral continuity of a (p, k)-Quasiposinormal operator and (p, k)- Quasihyponormal operator is investigated. It was proved that the (p, k)-Quasiposinormal operator is a pole of the resolvent set of the adjoint of the operator. It was also proved that if $\{T_n\}$ is a sequence of operators in the class of (p, k)-Quasiposinormal operators which converge in the operator's norm topology of an operator T in the same class, then the functions spectrum, Weyl spectrum, Browder spectrum and essential surjectivity spectrum are continuous at T. In [50] they proved that if T is (p, k)- quasiposinormal and $\bar{\lambda} \in \pi_{00}(T^*)$, then $\widetilde{T}$

is a pole of the resolvent set of T^*. They showed that if the spectrum is continuous at $T^* \in B(H)$ then the spectrum is continuous at T. They further proved that if $\{T_n\}$ is a sequence in (p, k)- quasiposinormal which converges in norm to T, then its spectrum is continuous at T and T^* is a point of continuity of σ_{ea}. Senthilkumar and Kiruthika in their work [49],[50], investigated continuity property of (p, k)- Quasiposinormal and (p, k)-Quasihyponormal operators and not the spectrum of posinormal operators. Some properties of posinormal operators like continuity and linearity have been investigated but its spectrum has not been fully investigated. In this study we shall investigate and characterize the spectrum of posinormal operators in an infinite dimensional complex Hilbert space.

Shapiro [47] investigated the relationship between the numerical range and the spectrum. Major points on this topic were: containment of the spectrum in the closure of the numerical range of an operator and the assertion that the intersection of the closures of the numerical ranges of all operators similar to an operator T gives precisely the convex hull of the of the spectrum of the operator T. Shapiro further gave an important proposition that the numerical range of an operator T contains all the eigenvalues of T. It was proved that the convex hull of the spectrum of an operator lies in the closure of the numerical range and that the numerical range is always convex (the Toeplitz- Hausdorff theorem). Shapiro's result obtained a complete description of the numerical ranges of 2×2 matrices; they are (possibly degenerate) elliptical discs with foci at the eigenvalues of the matrix. Shapiro investigated the relationship between the numerical range and spectrum of other operators but not that of posinormal operators. Meng [39] gave an important remark that if an operator T in a complex Hilbert space is normal, then the closure of the numerical range of the operator T is the smallest closed convex set containing the spectrum of T and the numerical range of the normal operator T is closed, the extreme points of the numerical range of T are eigenvalues. In [56] utility of quadratic numerical ranges and block numerical ranges were compared by Tretter. The convexity of the numerical range was found to be a useful property in the localization of the spectrum. Moreover, Tretter [56] introduced an alternative and sure method of localization of the spectrum by use quadratic numerical range. Tretter [56] investigated localization of the spectrum, description and structure of essential spectrum, block diagnalization, invariant subspaces, and the structure and utility of quadratic numerical ranges but not the numerical range of posinormal operators. In this study we have determined the numerical range of a posinormal operator and its spectrum. Moreover we have established the relationship between the numerical range and spectrum of a posinormal operator.

1.2 Basic concepts

In this section, we state the basic definitions and concepts on an operator, self- adjoint operator, normal operator, positive operator, posinormal operator, norm, inner product, numerical range, numerical radius, spectrum, spectral radius, which are useful in our study.

Definition 1.1 (23)**.** Let V, W be vector spaces over $\mathbb{K}$.

(i). A map $T : V \to W$ is called a linear tranformation if, for all $x, y \in V$

and $\alpha, \beta \in \mathbb{K}$, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

(ii). If $V = W$ then, a structure preserving map $T : V \to V$ is called a linear operator if, for all $x, y \in V$ and $\alpha, \beta \in \mathbb{K}$, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

Definition 1.2 (31)**.** A norm is a nonnegative real valued function taking $\|.\| : V \to \mathbb{R}$ such that $\forall\ x, y \in V$ and $\alpha \in \mathbb{K}$ the following axioms are satisfied:

(i). $\|x\| \geq 0$.

(ii). $\|x\| = 0$, if and only if $x = 0$.

(iii). $\|\alpha x\| = |\alpha|\|x\|$.

(iv). $\|x + y\| \leq \|x\|\|y\|$

The ordered pair $(\mathbb{V}, \|.\|)$ is called a normed space.

Definition 1.3 (31)**.** A Banach space is a complete normed space.

Definition 1.4 (23)**.** An inner product on a vector space V is a map $\langle ., . \rangle : V\ X\ V \to \mathbb{K}$ such that $\forall\ x,\ y,\ z \in V$ and $\lambda \in \mathbb{K}$; :

(i). $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$, if and only if $x = 0$.

(ii). $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(iii). $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$.

(iv). $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

The ordered pair $(V, \langle ., . \rangle)$ is called an inner product space.

Definition 1.5 (23)**.** A Hilbert space H is a complete inner product space.

Remark 1.6 (54)**.** Every Hilbert space is a Banach space but the converse is not necessarily true. The following are examples of Hilbert spaces:

Examples of Hilbert spaces

1. $\mathbb{C}^n$; an n-dimensional complex space is a Hilbert space whose norm is defined by:
$$\langle x, y\rangle = \sum_{k=1}^{n} x_k \overline{y}_k, \; \|x\| = (\sum_{k=1}^{n} |x_k|^2)^{\frac{1}{2}}.$$

2. l^2; space of converging sequences is a Hilbert space whose norm is defined by
$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \overline{y}_k, \; \|x\| = (\sum_{k=1}^{\infty} |x_k|^2)^{\frac{1}{2}}.$$

Definition 1.7 (31)**.** Let H be a Hilbert space and $T : H \to H$ be a linear operator. Then the operator T^* is called the adjoint of T defined by $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$

Definition 1.8 (31)**.** Let H be a Hilbert space. An operator $T : H \to H$ is called:

(i) Self adjoint/ Hermitian if $T = T^*$

(ii) Normal if $T^*T = TT^*$, that is, if T commutes with its adjoint.

(iii) Unitary if $T^*T = TT^* = I$

Definition 1.9 (31)**.** A positive operator T (denoted as $T \geq 0$) is a self adjoint operator such that $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

Definition 1.10 (44)**.** Let $T \in B(H)$. T is said to be posinormal if there exists a positive operator $P \in B(H)$ such that $TT^* = T^*PT$; where P is called the interrupter. $P(H)$ denotes the set of all posinormal operators on H. T is said to be coposinormal if T^* is posinormal.

Definition 1.11 (31)**.** Spectrum of an operator $T \in B(H)$ is the set $\sigma(T) = \{\lambda : T - \lambda I \text{ is not invertible}\}$. $\lambda \in \mathbb{C}$ is an eigenvalue of T if there exist $x \in H\setminus\{0\}$ such that $Tx = \lambda x$. $\mathbf{x} \neq 0$ is called the eigenvector of T corresponding to λ .

Definition 1.12 (31)**.** The resolvent set of an operator T is the compliment of the spectrum $\sigma(T)$ in the complex plane $\mathbb{C}$. It is denoted by $\rho(T)$.

Definition 1.13 (31)**.** Spectral radius $\gamma(T)$ of an operator T on H is given by :
$\gamma(T) := sup\{|\lambda|, \lambda \in \sigma(T)\}$.

Definition 1.14 (51)**.** Numerical range $W(T)$ of an operator T is the subset of the complex number $\mathbb{C}$ given by: $W(T) = \{\langle Tx, x\rangle, x \in H, \|x\| = 1\}$ with the following properties:

(i) $W(\alpha I + \beta T) = \alpha + \beta W(T)\ \forall\ \alpha, \beta \in \mathbb{C}$

(ii) $W(T^*) = \{\overline{\lambda}, \lambda \in W(T)\}$ where T^* is the adjoint operator of T.

(iii) $W(U^*TU) = W(T)U$. for any unitary operator U.

Remark 1.15 (56)**.** An important use of $W(T)$ is to bound the spectrum $\sigma(T)$ of the operator T. This is important in establishing the relationship between the numerical range and spectrum of posinormal operator.

Definition 1.16 (51)**.** Numerical radius $r(T)$ of an operator T on H is given by :
$r(T) = sup\{|\lambda| : \lambda \in W(T)\}$
$= sup\{| < Tx, x > |, \|x\| = 1\}$
with the following properties :

(i) $r(|T|) = \|T\|$.

(ii) $r(T^*T) = r(TT^*)$.

(iii) $r(UTU^*) = r(T)$.

Definition 1.17 (31)**.** An operator $T : H \to H$ is said to be bounded if there exist a constant $M > 0$ such that: $|Tx| \leq M|x|$.

Definition 1.18 (31)**.** Two vectors $x, y \in H$ are called orthogonal if $\langle x, y \rangle = 0$. It is denoted by $x \perp y$

Definition 1.19 (31)**.** A linear functional on a vector space V is an operator $T : V \to \mathbb{V}$, which satisfies the following properties

1. $T(x + y) = T(x) + T(y)\ \forall\ x, y \in V$ and

2. $T(\alpha x) = \alpha T(x)$, $\forall\ x, y \in V$ and $\alpha \in \mathbb{K}$.

1.3 Statement of the problem

Numerical ranges and spectra of Hilbert space operators like normal, hyponormal and self- adjoint operators have been characterized. However, for posinormal operators, other properties like continuity and linearity have been studied but not much has been investigated on the numerical range and the spectrum of posinormal operators. Therefore in this study, we have investigated the numerical range of posinormal operators, the spectrum of posinormal operators and lastly we have established the relationship between the numerical range and spectrum of a posinormal operator.

1.4 Objectives of the study

The objectives of the study have been to:

(i). Investigate numerical ranges of posinormal operators.

(ii). Investigate spectra of posinormal operators.

(iii). Establish the relationship between the numerical range and spectrum of a posinormal operator.

1.5 Research questions

(i). What are the numerical ranges of posinormal operators?

(ii). What are the spectra of posinormal operators?

(iii). Is there a relationship between the numerical range and spectrum of a posinormal operator?

1.6 Significance of the study

Hilbert space operators have been investigated by many researchers being motivated by the needs of operator theory, functional analysis and quantum theory. The numerical range and spectrum of posinormal operators have not been exhaustively investigated. The numerical ranges and spectra of posinormal operators have exemplary applications especially in quantum information theory, particularly, to optimize minimal output entropy of quantum channel, to detect entanglement using positive maps and for local distinguishability of unitary operators.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

In this chapter we review related literature for numerical ranges and spectra of posinormal operators and the relationship between numerical range and the spectrum of posinormal operators. Various studies have been carried out on the numerical range of normal operators, numerical range of positive operators, the spectrum of normal, self-adjoint, and positive operators.

2.2 Numerical Ranges

The concept of the numerical range (also called a field of values) since its introduction by Toeplitz [55], has remained to be a useful tool for studying matrices and operators. This has motivated extensive research on its properties and their generalizations. Toeplitz [55] proved that the numerical range of a matrix contains all its eigenvalues and the boundary of the numerical range is always a convex curve. Hausdorff [23] showed that the numerical range for general bounded linear operators is convex . Hausdorff further proved that for $T \in B(H)$ and H a complex Hilbert space, the spectrum of T is contained in the closure of the numerical range of T (that is, $\sigma(T) \subset \overline{W(T)}$). These two mathematicians developed the Toeplitz-Hausdorff theorem:

Theorem 2.1 (51, Theorem 3.2)**.** {*Toeplitz-Hausdorff theorem*} *The numerical range of every bounded linear operator T on a Hilbert space is convex. That is, for all $T \in B(H)$, $W(T)$ is convex.*

Proof. See [47] and [51]. □

This was true for all operators. Thus the convexity of the numerical range holds for posinormal operators. The theorem is the most essential result about numerical ranges and has many applications.

In [47] Shapiro proved the Toeplitz-Hausdorff theorem. The numerical range for 2×2 matrices was determined. Shapiro proved the Folk theorem:

Theorem 2.2 (51, Theorem 3.4)**.** {*Folk theorem*} *If $\lambda \in W(T)$ is a boundary point at which $\partial W(T)$ has infinite curvature then λ is an eigenvalue of T.*

Proof. See [47] and [51]. □

In [47] the numerical ranges of operators on two dimensional Hilbert space were characterized. Shapiro further worked on the Ellipse Theorem and gave a complete proof of it.

Theorem 2.3. {*The Ellipse theorem*} *If T is a linear transformation on $\mathbb{C}^2$, then $W(T)$ is a elliptical disc.*

Proof. See [47]. □

Shapiro investigated numerical ranges of operators on two dimensional Hibert space but not the numerical range of posinormal operators.

Hwa-Long and Pei [24] had excursions in numerical ranges. Anderson's condition for the numerical range of a finite matrix to equal a circular disc was studied. They surveyed Holbrook's conjecture on the numerical radius inequality concerning the product of two community operators. Lastly Hwa-Long and Pei investigated Williams and Crimmins's structure theorem on an operator when its numerical radius equals half of its norm. The excursion concentrated on investigations concerning the classical numerical ranges of operators and finite matrices. The following are their main results:

Theorem 2.4 (24, Theorem 3.1)**.** *If T is a $n \times n$ matrix such that $W(T)$ is contained in a closed elliptic disc E and there boundaries $\partial W(T)$ and ∂E intersect at more than n points, then $W(T) = E$.*

Proof. See [24]. □

Theorem 2.5 (24, Theorem 3.4)**.** *Let T be an $n \times n$ $(n \geq 3)$ matrix. Then*

(a). $\partial W(T)$ can contain at most $n - 2$ arcs of any ellipse, and

(b). If $W(T)$ contains an elliptic disc E and $\partial W(T)$ and ∂E intersect at more than n points, then $\partial W(T)$ contains an arc of ∂E. In this case, both numbers "$n-2$" and "n" are sharp.

Proof. See [24]. □

Theorem 2.6 (24, Theorem 4.1)**.** *If T and S doubly commute, then $W(TS) \leq min\{\|T\|W(S), W(T)\|S\|\}$*
In particular if this is the case if T or S is normal and $TS = ST$.

Proof. See [24]. □

Theorem 2.7 (24, Theorem 5.1)**.** *Let T be an operator on H such that $\|Tx\| = \|T\|$ for some unit vector $x \in H$. If $W(T) = 1$ and $\|T\| = 2$, then T is unitarily equivalent to an operator of the form $\begin{pmatrix} o & 2 \\ 0 & 0 \end{pmatrix} \oplus T$ and $W(T) = \overline{\mathbb{D}}$*

Proof. See [24]. □

While studying normality and the numerical range of an operator Stampfli [52] generalized results of Donoghue [16], Hildebrandt, Meng and Putnam [42]. He studied the convexity of a curve in the complex plane and remarked that if, at every point, the curve and the origin lie on the same side of the support line, the curve is said to be convex with respect to the origin. The main results are as below:

Theorem 2.8 (52, Theorem 1)**.** *Let $\sigma(T)$ lie on a curve $C \in \mathbb{C}$. Then T is normal if and only if $W(T^{\pm 1}) \subset \Sigma(T^{\pm 1})$.*

Proof. See [52]. □

Theorem 2.9 (52, Theorem 2)**.** *Let $\sigma(T)$ lie on a smooth convex curve. If (1) $W(T) \subset \Sigma(T)$ and (2) $W[(T-zI)^{-1}] \subset \Sigma[(T-zI)^{-1}]$, for z not a member of $\sigma(T)$, then T is normal.*

Stampfli studied numerical ranges of normal operators but not numerical ranges of posinormal operators.

In [18] Embry studied the numerical range of an operator on a complex Hilbert Space and came up with the following results:

Theorem 2.10 (18, Theorem 1)**.** *If $z \in W(A)$, then $YM_2 = M_2 \oplus M_2$ and*

(i). z is an extreme point of $W(A)$ if and only if M_2 is linear;

(ii). if z is a non extreme boundary point of $W(A)$, then YM_2 is a closed linear subspace of X and $YM_2 = \cup\{M_w : w \in L\}$, where L is the line of support of $W(A)$, passing through z. In this case $YM_2 = X$ if and only if $W(A) \subset L$.

Proof. See [18]. □

Theorem 2.11 (18, Theorem 2)**.** *Let $z \in W(A)$ and $K_z = \cap$ Maximal linear subspaces of M_2. If z is a boundary point of $W(A)$, let $N = \cup\{M_w | w \in L\}$, where L is a line of support for $W(A)$, passing through z.*

(i). If z is a boundary point of $W(A)$, $x \in K_z$ and $Ax \in N$, then $Ax = zx$ and $A^*x = z^*x$. Conversely, if $Ax = zx$ and $A^*x = z^*x$, then $x \in K_z$.

(ii). If $W(A)$ is a convex body and z is in the interior of $W(A)$, $K_z \equiv \{x|Ax = zx and A^*x = z^*x\}$.

Proof. See [18]. □

Embry investigated numerical ranges of other Hilbert space operators but not the numerical range of posinormal operators.

Skoufranis [51] also proved the Toeplitz-Hausdorff theorem and the Folk theorem. Skoufranis studied numerical ranges and developed various notions of numerical ranges of operators. These are the main results:

Theorem 2.12 (51, Theorem 1.7)**.** *Let $T \in B(H)$ be a normal operator. Then $r(T) = \|T\|$.*

Proof. See [51]. □

Skoufranis studied numerical ranges of Hilbert space like normal operators, self adjoint and unitary operators but not posinormal operators.

In [4] Barraa studied the numerical range of elementary operators and gave some results on the essential numerical range of the restriction of an elementary operator to the class of Hilbert- Schmidt. The following are the main results obtained:

Lemma 2.13 (4, Lemma 2.1)**.** *Let $T \in B(H)$. Each of the following conditions is necessary and sufficient in order that $\lambda \in V_e(T)$*

(1) $\langle Tx_n, x_n\rangle \to \lambda$ for some sequence $\{x_n\}$ of unit vectors such that $x_n \to 0$ weakly.

(2) $\langle Te_n, e_n \rangle \to \lambda$ *for some orthonormal sequence* $\{e_n\}$.

Theorem 2.14 (4, Theorem 2.2)**.** *Let* H, K *be two separable Hilbert spaces and* $A = (A_1, ..., A_p)$, $B = (B_1, ..., B_p)$ *two* p*- tuples with* $A_i \in B(H)$ *and* $B_i \in B(K)$ *for* $i = 1, ..., p$. *Then* $Co[(W_e(A) \circ W(B)) \cup (W(A) \circ W_e(B))] \subseteq V_e(R_2, A, B)$.

Proof. See [4]. □

Lemma 2.15 (4, Lemma 3.1)**.** *Let* A *be a non-negative, self-adjoint operator and* $AB = BA$. *Then* $V_e(AB) \subseteq V_e(A)V_e(B)$.

From the above literature review on numerical ranges it is clear that numerical ranges for various operators on a Hilbert space have been established but not for posinomal operators.

2.3 Spectrum

The study of the spectra of linear bounded operators on a Hilbert space has an extensive history. In [44] Rhaly introduced the concept of a posinormal operator. Areas of concern were self adjoint operators, normal operators, hyponormal operators, cohyponormal operators, seminormal and subnormal operators of all bounded linear operators on a Hilbert space H. He characterized an operator $T \in B(H)$, which is both positive ($\langle Tx, x \rangle \geqslant 0$) and normal ($TT^* = T^*T$). If $T \in B(H)$ is to be normal and positive, there must exist an interupter $P \in B(H)$ such that

$TT^* = T^*PT$, moreover T must be self adjoint. This result defines posinormality. The following theorems by Rhaly summarizes the most useful results he obtained:

Theorem 2.16 (44, Theorem 2.1)**.** *For $T \in B(H)$ the following statements are equivalent:*

(1) T is posinormal

(2) $RanT \subseteqq RanT^*$

(3) $TT^* \leqq \lambda^2 T^*T$ for some $\lambda \geqq 0$; and

(4) There exists a $S \in B(H)$ such that $T = T^*S$. Moreover if [1.],[2.],[3.] and[4.] hold, there is a unique operator S such that:

(a) $\|S\|^2 = \inf[\mu : TT^* \leqq \mu T^*T]$;

(b) $KerT = KerS$ and

(c) $RanS \subseteqq (RanT)^-$

Theorem 2.17 (44, Theorem 3.1)**.** *Every invertible operator is posinormal.*

Proof. See [44]. □

In [27] Itoh gave a characterization of posinormal operators different from Rhaly's:

Theorem 2.18 (27, Theorem 2)**.** *An operator T is posinormal if, and only if, there exists $\lambda > 0$ such that $\mid T(T \mid x, y) \mid \leq \lambda \| \mid T \mid x \| \| \mid T \mid y \|$ for all $x, y \in H$.*

Proof. See [27] □

Itoh showed that if an operator T is posinormal then it is M-paranormal. Itoh came up with a class of operators which he called P-Posinormal operators, $(p - P(H))$, gave a characterization of P-posinormal operators and further proved that if an operator T is P-posinormal, then it is also M-paranormal. Itoh did not investigate the spectrum of posinormal operators.

In [50] they studied the (p,k)- quasiposinormal operators in relation to the spectral mapping theorem for Weyl spectrum, $\sigma_w(T)$. An operator T is said to be (p, k)- quasiposinormal if $T^{*k}(c^2(T^*T)^p - (TT^*)^p)T^k \geq 0$ for a positive integer $0 < p \leq 1$, $c > 0$ and a positive integer k. They defined Weyl spectrum by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl } \}.$$

They obtained the following important results:

(i) Weyl's theorem holds for (p, k)- quasiposinormal operator T for $c > 0$, that is, $\sigma(T) \backslash \sigma_w(T) = \pi_{00}(T)$ where $\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$.

(ii) If T is a (p, k)- quasiposinormal operator, then $\sigma(T) - \{0\} = \sigma_e(T) - \{0\}$ where $\sigma_e(T)$ is the compression spectrum of T.

(iii) Weyl's theorem holds for T if its adjoint T^* is a (p, k)- quasiposonormal operator.

In [49] spectral continuity of (p, k)- quasiposinormal operator and (p, k)- quasihyponormal operator is investigated. They proved that if T is (p, k)- quasiposinormal and $\bar{\lambda} \in \pi_{00}(T^*)$, then T is a pole of the resolvent set of T^*. They showed that if the spectrum is continuous at $T^* \in B(H)$ then the spectrum is continuous at T. They further proved that if $\{T_n\}$ is a sequence in (p, k)- quasiposinormal which converges in norm to T, then its spectrum is continuous at T and T^* is a point of continuity of σ_{ea}.

Duggal and Kubrusly [17] studied Weyl's theorems for posinormal operators. They worked on Conditionally Totally Posinormal (CTP for short) operators and Totally Posinormal (TP for short) operators. A posinormal operator T is said to be Conditionally totally posinormal ($T \in CTP$) if to each complex number λ there corresponds a positive operator P_λ such that $|(T-\lambda I)^*|^2 = |P_\lambda^{\frac{1}{2}}(T-\lambda)|^2$ for all λ and $T \in B(H)$ is said to be totally posinormal ($T \in TP$) if to each complex number λ there corresponds a positive operator P such that $|(T-\lambda I)^*|^2 = |P^{\frac{1}{2}}(T-\lambda)|^2$ for all λ. Duggal and Kubrusly remarked that the restriction of a posinormal operator to an invariant subspace is again posinormal. A definition of posinormal operators equivalent to that of Rhaly was given [7] that $T \in B(H)$ is posinormal if there exists a co- isometry $V^* \in B(H)$ and a positive operator $P \in B(H)$ such that $T = T^*PV^*$. Duggal and Kubrusly observed that the large class of posinormal operators contains other classes such as the classes consisting of hyponormal operators ($T \in B(H) : TT^* \leq T^*T$); M-hyponormal operators ($T \in B(H) : |(T-\lambda I)^*|^2 \leq M|(T-\lambda I)|^2$) for some real number $M > 0$ and dominant operators ($T \in B(H) : |(T - \lambda I)^*|^2 \leq M_\lambda |(T - \lambda I)|^2$) for some real number $M_\lambda > 0$ and all complex numbers λ. The following theorems give the main results they obtained:

Theorem 2.19 (17, Theorem 3.1)**.** *Let $T \in TP$. Then*

(1) $f(T)$ and $f(T^*)$ satisfy Weyl's theorem for every $f \in H(\sigma(T))$.

(2) T^* satisfies a- Weyl's theorem.

(3) If T^* has the single valued extension property (SVEP), then T satisfies a- Weyl's theorem.

In [46] Rhaly introduced a superclass of the posinormal operators and gave sufficient conditions for an operator in this superclass to be posinormal and hyponormal. A clear proof of hyponormality of the generalized Cesàro operator of order one (C_k) was given. The following three theorems gives a summary of Rhaly's main results:

Theorem 2.20 (46, Theorem 1)**.** *Suppose $A \in B(H)$ satisfies $AQA^* = A^*PA$ or positive operators $P, Q \in B(H)$.*

(a) If Q has dense range, then A is supraposinormal and $KerA \subset KerA^*$.

(b) If P has dense range, then A is supraposinormal and $KerA^* \subset KerA$.

(c) If Q is invertible, then the supraposinormal operator A is posinormal.

(d) If P is invertible, then the supraposinormal operator A is coposinormal.

(e) If P and Q are both invertible, then A is both posinormal and coposinormal with $KerA = KerA^*$ and $RanA = RanA^*$.

Proof. See [46]. □

Theorem 2.21 (46, Theorem 2)**.** *Assume $A - \lambda$ is supraposinormal for distinct real values $\lambda = 0$, r_1 and r_2 and assume that the same interrupter pair (Q, P) serves $A - \lambda$ in each of those three cases. Then $Q = P$ and $Ker(A - \lambda) = Ker(A - \lambda)^*$ when $\lambda = 0$, r_1 and r_2.*

Proof. See [46]. □

Theorem 2.22 (46, Theorem 3)**.** *If $A \in B(H)$ is totally supraposinormal and the same two positive operators $Q, P \in B(H)$ form an interrupter pair (Q, P) for $A - \lambda$ for all complex numbers λ, then $Q = P$; it follows that $Ker(A - \lambda) = Ker(A - \lambda)^*$ for all λ.*

Proof. See [46]. □

Rhaly investigated continuity and invertibility properties of posinormal operators and gave sufficient conditions for an operator to be posinormal but did not investigate the spectrum of posinormal operators. Evidently spectral properties for several Hilbert space operators have been studied and a great deal of generazations given but not for posinormal operators.

2.4 Relationship between numerical ranges and the spectrum of operators

Meng [39] remarked that if T is normal, the closure of $W(T)$ is the smallest closed convex set containing the spectrum of T and that if T is normal and

$W(T)$ is closed, the extreme points of $W(T)$ are eigenvalues. Hildebrandt had come up with an important result that, for a bounded linear operator T on a Hilbert space, you get the convex hull of $\sigma(T)$ by intersecting the closures of the numerical ranges of all the operators similar to T [2]. Shapiro [47] gave a short and complete proof to the Hildebrandt's theorem. The well known results [21] that $r(T) \leq \|T\|$ for any operator T on a Hilbert space, and that the spectral radius is a similarity invariant were used to show that $r(T) \leq inf\{\|VTV^{-1}\| : V \text{ is invertible on } H\}$ thus suggesting a connection between the spectrum and the numerical range. Shapiro proved the result that the spectrum of an operator lies in the closure of its numerical range. The following theorems give useful results:

Theorem 2.23 (47, Theorem 5.1)**.** *If T is a bounded linear operator on a Hilbert space H, then T is contained in the closure of the numerical range of T.*

Proof. See [47]. □

Proposition 2.24 (47, Proposition 1.1)**.** *$W(T)$ contains all of the eigenvalues of T.*

Shapiro investigated the relationship between the numerical range and spectrum of normal operators but not posinormal operators.

In [28] matrices for which the numerical range coincides with the convex hull of the spectrum were investigated. A key observation was made that the eigenvectors corresponding to any eigenvalue occurring on the boundary of the numerical range must be orthogonal to eigenvectors corresponding to all other eigenvalues. The following theorems are some of the results obtained by Johnson:

Theorem 2.25 (29, Theorem 1)**.** *Suppose* $\alpha \in \partial W(A) \bigcap \sigma(A)$ *and* $Ax = ax$, $x^*x = 1$ *we have*

(i) If α *has algebraic multiplicity* m *in* $\sigma(A)$*, then the dimension of the eigen space for* α *is* m*;*

(ii) For any $\alpha \neq \lambda \in \sigma(A)$*,* $Ay = \lambda y$*,* $y^*y = 1$*, it follows that* $x^*y = 0$*; and*

(iii) A *is unitarily equivalent to* $\alpha I \oplus B$ *where* α *is not a member of* $\sigma(B)$*.*

Proof. See [29]. □

Theorem 2.26 (29, Theorem 2)**.** A *is normal if and only if* A *is unitarily equivalent to* $A_1 \oplus \dots\dots\dots \oplus A_k$*, where* A_i *satisfies the boundary property,* $i = 1, \dots\dots k$*.*

Proof. See [29]. □

Theorem 2.27 (29, Theorem 3)**.** *We have* $W(A) = Co(\sigma(A))$ *if and only if* A *is normal or* A *is unitarily equivalent to* $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$*, where* A_1 *is normal and* $W(A_2) \subseteq W(A_1)$*.*

Proof. See [29]. □

In describing the relationship between the numerical range and spectrum of linear operators Skoufranis [51] obtained the following results:

Theorem 2.28 (51, Theorem 2.9)**.** *Let $T \in B(H)$. Then $\sigma(T) \subseteq \overline{W(T)}$.*

Proof. See [51]. □

Theorem 2.29 (51, Theorem 2.11)**.** *Let $T \in B(H)$ be a normal operator. Then $\overline{W(T)} = Conv(\sigma(T))$.*

Proof. See [51]. □

Theorem 2.30 (51, Theorem 5.10)**.** *Let $T \in B(H)$. Then $\sigma_e(T) \subseteq W_e(T)$.*

Proof. See [51]. □

Theorem 2.31 (51, Theorem 5.11)**.** *Let $T \in B(H)$ be a normal operator on an infinite dimensional Hilbert space H. Then $W_e(T) = Conv(\sigma_e(T)$.*

Proof. See [51]. □

Skoufranis investigated the relationship between the numerical range and spectrum of normal operators but not the relationship between the numerical range and spectrum of a posinormal operator.

In [56] Tretter worked on localization of the spectrum, description of the essential spectrum, investigation of its structure, block diagnalization and invariant subspaces. Utility of numerical ranges and quadratic numerical ranges were compared. Tretter noted that the convexity of the numerical range is a useful property for localization of the spectrum since the spectrum lies in half plane. Tretter introduced an alternative way of localizing the spectrum by use of quadratic numerical range.

From the literature review we see that the relationship between the spectrum and the numerical range of normal and linear operators has been established but the relationship between the spectrum and numerical range of posinormal operators has not been established. In this study we shall establish the relationship between the numerical range and the spectrum of a posinormal operator.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

For a successful completion of this research, background knowledge of Functional analysis, the operator theory, especially normal operators, self-adjoint operators, hyponormal operators on a Hilbert space, numerical range and the spectrum of operators on a Hilbert space is vital. We have stated some known fundamental principles which were useful in our research. The methodology involved the use of known inequalities and techniques like Cauchy -Schwartz inequality and the polarization identity.Lastly, we have used the technical approach of tensor products in solving the stated problem.

3.2 Fundamental Principles

The following fundamental principles have been useful in our work:

Theorem 3.1 (51, Theorem 3.2)**.** $\{$*The Toeplitz- Hausdorff Theorem*$\}$ *The numerical range of every bounded linear operator* T *is convex.*

Theorem 3.2 (51, Theorem 3.4)**.** $\{$ *The Folk theorem*$\}$ *Let* $T \in B(H)$ *be such that* $\lambda \in \delta W(T)$*. If no closed disc of* $W(T)$ *contains* λ*, then* λ *is an eigenvalue of* T*.*

Theorem 3.3. *Let* $T \in B(X, X)$*. If* $\|T\| < 1$ *then* $(I - T)^{-1}$ *exists as a bounded operator on* X *and* $(1 - T)^{-1} = \sum_{j=0}^{\infty} T^j = I + T + T^2 + ...$

Theorem 3.4. *The resolvent set* $\rho(T)$ *of a bounded linear operator* T *on a complex Banach space* X *is open, implying that the spectrum of* T *is closed.*

Theorem 3.5. *The spectrum* $\sigma(T)$ *of a bounded self-adjoint linear operator* $T : H \to H$ *on a complex Hilbert space* H *is real.*

Theorem 3.6. *Let* $T \in B(H)$ *be a self adjoint operator. Then the numerical radius of* T *is equal to the norm of* T*.*

Theorem 3.7. *Let* $T \in B(H)$ *be a normal operator. Then* $r(T) = \|T\|$*.*

3.3 Known Inequalities and Techniques

We have used the following inequality and identity in our work:

Definition 3.8. Cauchy - Schwarz inequality; If V is an inner product space and $\|v\| = \sqrt{\langle v, v\rangle}\ \forall\ v \in V$ then: $|\langle x, y\rangle| \leq \|x\|\ \|y\|\ \forall\ x,\ y \in V$

Theorem 3.9. ***Polarization Identity;*** *Let X be an inner product space, then for arbitrary $x, y \in X$, $\langle x, y \rangle = \frac{1}{4}[\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2]$.*

Our technical approach involved use of tensor product of operators. This helped us to study the properties of posinormal operators. The following background knowledge is important in this research:

Tensor product of operators

For every pair of linear operators T on V and S on W, there exists a unique linear operator $T \otimes S$ on $V \otimes W$ such that $(T \otimes S)(v \otimes w) = (Tv)(Sw)$.

We use the theory of tensor product that will be useful in investigating the numerical ranges and spectra of posinormal operators, as well as, in establishing the relationship between the numerical range and spectrum of a posinormal operator.

Tensor product of Hilbert spaces

Construction

Let H and K be Hilbert spaces over $\mathbb{C}^2$. Let $H \otimes_{alg} K$ denote the tensor product as vector spaces, i.e

$$H \otimes_{alg} K = F/N,$$

where F is the vector space freely generated by $H \times K$ and N is the subspace spanned by elements of the form

$$(\alpha v_1 + v_2, w) - \alpha(v_1, w) - (v_2, w), (v, \alpha w_1 + w_2) - \alpha(v_1 w_1) - (v, w_2)$$

where $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $\alpha \in \mathbb{K}$. The scalar products $\langle ., . \rangle_H$ on H and $\langle ., . \rangle_K$ on K define a mapping $\widehat{s} : F \times F \to \mathbb{K}$ by

$$\widehat{s}(\varphi, \psi), ((\varphi\prime, \psi\prime)) := \langle \varphi, \varphi\prime \rangle_H \langle \psi, \psi\prime \rangle_K$$

and extension by antilinearity in the first argument and linearity in the second argument, that is
$\widehat{s}(\sum_i \alpha_i(\varphi_i, \psi_i), \sum_j \alpha\prime_j(\varphi\prime_j, \psi\prime_j)) = \sum_{ij} \alpha_i^* \alpha\prime_j \langle \varphi, \varphi\prime_j \rangle_H \langle \psi, \psi\prime_j \rangle_K$.
Antilinearity of $\langle ., . \rangle_H$ and $\langle ., . \rangle_K$ in the first argument implies

$$\begin{aligned}
\widehat{s}(N \times F) &= 0 : \widehat{s}(\alpha\varphi_1 + \varphi_2, \psi) - \alpha(\varphi_1, \psi) - (\varphi_2, \psi), (\varphi\prime, \psi\prime) \\
&= \langle \alpha\varphi_1 + \varphi_2, \varphi\prime \rangle_H \langle \psi, \psi\prime \rangle_K - \alpha^* \langle \varphi_1, \varphi\prime \rangle_H \langle \psi, \psi\prime \rangle_K - \langle \varphi_2, \varphi\prime \rangle_H \langle \psi, \psi\prime \rangle_K \\
&= 0,
\end{aligned}$$

and similarly for the other type of spanning vectors. By an analogous argument, linearity of $\langle ., . \rangle_H$ and $\langle ., . \rangle_K$ in the second argument implies $\widehat{s}(F \times N) = 0$. It follows that $\widehat{s}$ descends to a mapping

$$\langle ., . \rangle : (F/N) \times (F/N) \equiv (H \otimes K) \times (H \otimes K) \to \mathbb{C}, \langle [x], [y] \rangle := \widehat{s}(x, y),$$

which is antilinear in the first argument and linear in the second argument. We compute

$$\begin{aligned}
\langle (\varphi \otimes \psi), (\varphi\prime \otimes \psi\prime) \rangle &= \langle [(\varphi, \psi)], [(\varphi\prime, \psi\prime)] \rangle \\
&= \widehat{s}((\varphi, \psi), (\varphi\prime, \psi\prime)) \\
&= \langle \varphi, \varphi\prime \rangle_H \langle \psi, \psi\prime \rangle_K.
\end{aligned}$$

We claim that $\langle .,. \rangle$ is a scalar product and

1. $\langle [x],[y]\rangle = \langle [y],[x]\rangle_*$: by (anti)linearity, it suffices to check this for pure tensor products,

$$\begin{aligned}\langle \varphi\otimes\psi, \varphi\prime\otimes\psi\prime\rangle &= \langle \varphi,\varphi\prime\rangle_H\langle\psi,\psi\prime\rangle_K \\ &= \langle \varphi\prime,\varphi\rangle_H^*\langle\psi\prime,\psi\rangle_K^* \\ &= \langle \varphi\prime\otimes\psi\prime, \varphi\otimes\psi\rangle^*;\end{aligned}$$

2. $\langle [x],[x]\rangle = 0$ implies $[x] = 0$.

Now, having a scalar product on $H\otimes_{alg} K$, we can define the tensor product of Hilbert spaces $H\otimes K$ to be the completion of $H\otimes_{alg} K$ in the corresponding norm.

Theorem 3.10 (36). *Let H and K be Hilbert spaces. Then there exists a unique inner product $\langle .,. \rangle$ on $H\otimes K$ such that*

$$\langle x\otimes y, x'\otimes y'\rangle = \langle x, x'\otimes y, y'\rangle : x, x' \in H, y, y' \in K.$$

Lemma 3.11 (8). *Let H and K be Hilbert spaces and suppose that $u \in B(H)$ (a set of all bounded operators on H) and $v \in B(K)$ (a set of all bounded operators on K). Then there is a unique operator $u\widehat{\otimes}v \in B(H\widehat{\otimes}K)$ such that*

$$(u\widehat{\otimes}v)(x\otimes y) = u(x)\otimes v(y), x\in H, y\in K.$$

Furthermore, it holds that

$$\|u\widehat{\otimes}v\| = \|u\|\|v\|.$$

Theorem 3.12 (9)**.** *[8] The tensor product on the inner product space $H \otimes K$ of two operators $T \in B(H)$ and $S \in B(K)$ is the transformation $T \otimes S : H \otimes K \to H \otimes K$ of $H \otimes K$ into itself such that*

$$T \otimes S \sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n} Tx_i \otimes Sy_i$$

for every $\sum_{i=1}^{n} x_i \otimes y_i \in H \otimes K$.

Example 3.13 (1)**.** Let (X, A, μ) be a measure space. Clearly, $L^2(X, A, \mu)$ is the space of real valued measurable functions that are square integrable. Now, we consider the case that $X \subseteq R^n$ and A is the Borel $\sigma-$algebra on X.

Therefore, we drop the $\sigma-$algebra and use the notation (L^2X, μ) for brevity. Suppose $\{\phi_i(x)\}$ and $\{\eta_i(y)\}$ are orthonormal bases for $L^2(\varrho_1, \mu_1)$ and $L^2(\varrho_2, \mu_2)$ respectively. Consider then the set $\{\phi_i(x)\eta_j(y)\}$. We can show that $\{\phi_i(x)\eta_j(y)\}$ is a complete orthonormal set in $L^2(\varrho_1 \times \varrho_2, \mu_1 \otimes \mu_2)$ as follows:

First note that orthonormality is immediate. We proceed to prove completeness. Let $f(x, y) \in L^2(\varrho_1 \times \varrho_2, \mu_1 \times \mu_2)$ be such that

$$\int_{\varrho_1} \int_{\varrho_2} f(x, y)\phi_i(x)\eta_j(y)d\mu_1(x)d\mu_2(y) = 0,$$

for all i,j. We claim that $f = 0$ (almost everywhere). We have ,

$$0 = \int_{\varrho_1}\int_{\varrho_2} f(x,y)\phi_i(x)\eta_j(y)d\mu_1(x)d\mu_2(y) = \int_{\varrho_2}(\int_{\varrho_1} f(x,y)\phi_i(x), d\mu_1(x))\eta_j(y)d\mu_2(y).$$

where the second equality follows by Fubini's theorem. Now using the fact that $\{\eta_j\}$ is orthonormal basis for $L^2(\varrho_2, \mu_2)$ we have that for all i

$$\int_{\varrho_1} f(x,y)\phi_i(x), d\mu_1(x) = 0,$$

μ_2 almost everywhere. Let $E_i \subseteq \varrho_2$ be set of measure zero where the above equality does not hold and let $E = \cup_i E_i$. Then, for $y \neq E$,

$$\int \varrho_1 f(x,y)\phi_i(x), d\mu_1(x) = 0$$

for all i, and thus, again using the fact that $\{\phi_i\}$ is a complete orthonormal set in $L^2(\varrho_1, \mu_1)$, we have that $f(x,y) = 0$, μ_1^* almost everywhere. Therefore, $f = 0$, $(\mu_1 \otimes \mu_2)^*$ almost everywhere. Hence, we have shown that $\{\phi_i(x)\eta_j(y)\}$ is a complete orthonormal set in $L^2(\varrho_1 \times \varrho_2, \mu_1 \otimes \mu_2)$.
We can also show the isomorphism

$$L^2(\varrho_1 \times \varrho_2, \mu_1 \otimes \mu_2) \cong L^2(\varrho_1, \mu_1) \otimes L^2(\varrho_2, \mu_2)$$

as follows.
Let us define a mapping U that takes an orthonormal basis of $L^2(\varrho_1, \mu_1) \otimes L^2(\varrho_2, \mu_2)$ onto an orthonormal basis of $L^2(\varrho_1 \times \varrho_2, \mu_1 \otimes \mu_2)$:

$$U(\phi_i \otimes \eta_j) = \phi_i(x)\eta_j(y)$$

. Note that U extends uniquely to unitary mapping of $L^2(\varrho_1, \mu_1) \otimes L^2(\varrho_2, \mu_2)$ onto $L^2(\varrho_1 \times \varrho_2, \mu_1 \otimes \mu_2)$.
Moreover, note that for $f \in L^2(\varrho_1, \mu_1)$ and $g \in L^2(\varrho_2, \mu_2)$, $f = \sum_i c_i \phi_i$, $g = \sum_j d_j \eta_j$, and we have,

$$\begin{aligned} U(f \otimes g) &= U((\sum_i c_i \phi_i) \otimes (\sum_j d_j \eta_j)) \\ &= U(\sum_{i,j} c_i d_j \phi_i \eta_j) \\ &= (\sum_i c_i \phi_i)(\sum_j d_j \eta_j) = fg. \end{aligned}$$

Chapter 4

RESULTS AND DISCUSSIONS

4.1 Introduction

In this chapter we have discussed the main results on the numerical range of posinormal operators, the spectrum of posinormal operators and the relationship between the numerical range and spectrum of posinormal operators. In the first section we discuss numerical ranges of posinormal operators.

4.2 Numerical range

Lemma 4.1. *Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Let $A \in B(H)$ be posinormal then $W(A)$ is an ellipse whose foci are the eigenvalues of A.*

Proof. Choose A such that $A = \begin{pmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{pmatrix}$ with λ_1 and λ_2 as the eigenvelues of A. Now if $\lambda_1 = \lambda_2 = \lambda$, we have

$$A - \lambda I = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

Let $x = (x_1, x_2)$, then

$$(A - \lambda I)x = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_2 \\ 0 \end{pmatrix} = a \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$$

Therefore,

$$\|A - \lambda I\| = sup\{\|a(x_2, 0)\| : |x_1|^2 + |x_2|^2 = 1\} = |a|$$

. Hence the radius is $\frac{1}{2}|a|$. Therefore the numerical range

$$W(A) = \{z : |z| \leq \frac{|a|}{2}\}.$$

It thus follows that $W(A)$ is a circle with center at λ and radius $\frac{|a|}{2}$. Now if $\lambda_1 \neq \lambda_2$ and $a = 0$ we have $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. If $x = (x_1, x_2)$, then

$$Ax = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{pmatrix}.$$

Therefore taking the inner product $\langle Ax, x\rangle$ we get

$$\langle Ax, x\rangle = \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 \overline{x_1} + \lambda_2 x_2 \overline{x_2} \end{pmatrix} = \begin{pmatrix} \lambda_1 |x_1|^2 + \lambda_2 |x_2|^2 \end{pmatrix}.$$

So $\langle Ax, x\rangle = \lambda_1|x_1|^2 + \lambda_2|x_2|^2$. Now letting $t = |x_1|^2$, we therefore write the above equation as follows

$$\langle Ax, x\rangle = t\lambda_1 + (1-t)\lambda_2$$

, since $|x_1|^2 + |x_2|^2 = 1$. So $W(A)$ is the set of convex combinations of λ_1 and λ_2 and is the segment joining them. If $\lambda_1 \neq \lambda_2$ and $a \neq 0$ we choose λ such that it lies between λ_1 and λ_2. We therefore have

$$A - \frac{\lambda_1 + \lambda_2}{2} I = \begin{pmatrix} \frac{\lambda_1-\lambda_2}{2} & a \\ 0 & \frac{\lambda_2-\lambda_1}{2} \end{pmatrix}$$

In this case, we let $z = re^{-i\theta}$, $\frac{\lambda_1-\lambda_2}{2} = re^{-i\theta}$ and $\frac{\lambda_2-\lambda_1}{2} = -re^{-i\theta}$ so, $e^{-i\theta}\begin{pmatrix} A - \frac{\lambda_1+\lambda_2}{2} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1-\lambda_2}{2} & a \\ 0 & \frac{\lambda_2-\lambda_1}{2} \end{pmatrix} = A'$. Here we see that $W(A')$ is an ellipse with the center at $(0,0)$ and the minor axis $|a|$, and foci at $(r, 0)$ and $(-r, 0)$. Thus, $W(A)$ is an ellipse with foci at λ_1 and λ_2 and the major axis has an inclination of θ with the real axis. $\square$

Remark 4.2. Lemma 4.1 follows [41, Lemma 4] analogously.

Example 4.3 (41, Example 5)**.** Let $A \in \mathbb{C}^2$ be the operator defined by the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Take $x \in \mathbb{C}^2$, $x \in (f, g)$, $\|x\|^2 = |f|^2 +$

$|g|^2 = 1$ with $\|x\| = 1$ $Ax = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}$ and $\langle Ax, x \rangle =$ $\begin{pmatrix} g & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = g\overline{f}$. Taking absolute values on both sides we have

$$|\langle Ax, x \rangle| = |f||g| = \frac{1}{2}(|f|^2 + |g|^2) = \frac{1}{2}$$

So $W(A) \subset \{z : |z| \leq \frac{1}{2}\}$, a circle of radius $\frac{1}{2}$ centered at $(0, 0)$.

Alternatively, given the operator A defined by the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

We then have the characteristic polynomial given by

$$A - \lambda I = \begin{pmatrix} 0 - \lambda & 1 \\ 0 & 0 - \lambda \end{pmatrix}$$

and hence finding the characteristic equation we see that $\lambda^2 = 0$. Therefore, $\lambda = 0$ is the eigenvalue. Since for the norm we have $\frac{1}{2}\|A\|$ and therefore normalizing the vector x we see that $(\|\frac{x}{\|x\|}\|) = 1$. Now we have $A(f, g) = (g, 0)$, that is $A_x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}$. This implies that $\|A(f, g)\| = \|(g, 0\| = \|g\|$. From the definition of an operator norm,

$$\begin{aligned} \|A\| &= \sup\{\|A(f, g)\| : \|(f, g)\| = 1\} \\ &= \sup\{\|A(f, g)\| : \sqrt{f^2 + g^2} = 1\} \\ &= \sup\{\|g\| : f^2 + g^2 = 1\} \\ &= 1. \end{aligned}$$

Therefore, $\frac{1}{2}\|A\| = \frac{1}{2}(1) = \frac{1}{2}$. Therefore, $W(A)$ is a circle of radius $\frac{1}{2}$

centered at $(0,0)$.

Example 4.4 (41, Example 6)**.** Let A be the unilateral shift on l_2 of square summable sequences. For any $x \in l_2, x = (x_1, x_2, x_3, ...)$, with $\|x\| = 1$ and $\sum_{i=1}^{\infty} |x_i|^2 < \infty$, the unilateral right shift operator $A : l_1 \to l_2$ is given by $Ax = (0, x_1, x_2, x_3, ...)$.
Now,

$$\begin{aligned} \langle Ax, x\rangle &= \langle \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \rangle \\ &= 0\overline{(x_1)} + x_1\overline{x_2} + x_2\overline{x_3} + ... \\ &= x_1\overline{x_2} + x_2\overline{x_3} + ... \end{aligned}$$

Thus, $(|x_1| - |x_2|)^2 \geq 0$ by the arithmetic- geometric mean inequality implies that $|x_1|^2 + |x_2|^2 \geq 2(|x_1||x_2|)$. Similarly, $|x_2|^2 + |x_3|^2 \geq 2(|x_2||x_3|)$, also $|x_3|^2 + |x_4|^2 \geq 2(|x_3||x_4|)$ and so on. Therefore adding all the terms on the left and similarly on the right of the above equations, we obtain $|x_1|^2 + 2|x_2|^2 + 2|x_3|^2 + 2|x_4|^2 + ... \geq 2|x_1||x_2| + 2|x_2||x_3| + 2|x_3||x_4| + ...$
We therefore have

$$\begin{aligned} |\langle Ax, x\rangle| &\leq |x_1\overline{x_2}| + |x_2\overline{x_3}| + ... \\ &= |x_1||\overline{x_2}| + |x_2||\overline{x_3}| + ... \\ &= |x_1||x_2| + |x_2||x_3| + ... \\ &= \frac{1}{2}(2|x_1||x_2| + 2|x_2||x_3| + ...) \end{aligned}$$

Now since $\|x\| = |x_1|^2 + |x_2|^2 + ... = 1$, we have

$$
\begin{aligned}
|\langle Ax, x\rangle| &= \frac{1}{2}[|x_1|^2 + 2|x_2|^2 + 2|x_3|^2 + ...] \\
&= \frac{1}{2}[(|x_1|^2 + |x_2|^2 + |x_3|^2 + ...) + (|x_2|^2 + |x_3|^2 + ...)] \\
&= \frac{1}{2}[(1 + |x_2|^2 + |x_3|^2 + ...) \\
&= \frac{1}{2}[1 + (1 - |x_1|^2)] \\
&= \frac{1}{2}[2 - |x_1|^2]
\end{aligned}
$$

If $|x_1| \neq 0$ we see that $|\langle Ax, x\rangle| \leq 1$. For if $|x_1| = 0$ and x contains a finite number of nonzero entries, we have $|\langle Ax, x\rangle| = 1$ if we consider a minimum natural number n such that $x_n \neq 0$. Therefore, $W(A)$ is an open disc of radius < 1.

Lemma 4.5. *Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. If $A \in B(H)$ is posinormal, then $W(A)$ is nonempty.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be an orthornomal sequence of vectors in H. For $\{x_n\}_{n=1}^{\infty}$ to exist in H then

$$lim_{n\to\infty}\langle Ax_n, x_n\rangle = a.$$

The sequence $\{\langle Ax_n, x_n\rangle\}_{n=1}^{\infty}$ is bounded and $\|x\| = 1$ because x_n has norm 1. Now, using $A = A^*$ (because all posinormal operators are self adjoint by the fact that they are all positive operators) we have

$$\begin{aligned}
\langle Ax_n - \|A\|x_n, Ax_n - \|A\|x_n\rangle &= \langle Ax_n, Ax_n\rangle - \langle Ax_n, \|A\|x_n\rangle - \langle \|A\|x_n, Ax_n\rangle + \\
\langle \|A\|x_n, \|A\|x_n\rangle & \\
&= \|Ax_n\|^2 - 2\|A\|\langle Ax_n, x_n\rangle + \|A\|^2\|x_n\|^2 \\
&\leq 2\|A\|^2\|x_n\|^2 - 2\|A\|\langle Ax_n, x_n\rangle \\
&= 2\|A\|^2\|x_n\|^2 - 2\|A\|\langle Ax_n, x_n\rangle \\
&\Rightarrow 2\|A\|^2\|x_n\|^2 - 2\|A\|^2\|x_n\|^2 \\
&= 0
\end{aligned}$$

Therefore, as $n \to \infty$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to 0 in H such that

$$lim_{n\to\infty}\langle Ax_n, x_n\rangle = a.$$

Thus x is an eigenvector for the eigenvalue $\|A\|$. This implies that $W(A)$ is nonempty. □

The next result due to Toeplitz and Hausdorff [47, Theorem 4.1], shows that the numerical range of linear operators on a Hilbert space is always convex. We give its proof when $\lambda_1 = \lambda_2$, $\forall \lambda_1, \lambda_2 \in W(A)$ for completion.

Theorem 4.6. *Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Let $A \in B(H)$, then $W(A)$ is always convex.*

Proof. Let $\lambda_1, \lambda_2 \in W(A)$, $\lambda_1 = \lambda_2$. We prove that $(1-t)\lambda_1 + t\lambda_2 \in W(A)$ whenever $t \in [0, 1]$.

If $B = \alpha I + \beta A$, where $\alpha, \beta \in \mathbb{C}$ are such that $0 = \alpha + \beta\lambda_1$ and $1 = \alpha + \beta\lambda_2$

it is sufficient to show that $t \in W(B)$ for all $t \in [0,1]$. Let us fix unit vectors $x, y \in H$ such that $0 = (Bx|x)$, $1 = (By|y)$ and define $g : \mathbb{R} \to \mathbb{C}$ by $g(t) = (Bx|y)e^{(-it)} + (By|x)e^{(it)}$, $t \in \mathbb{R}$
Moreover, there exists $t_0 \in [0, \pi]$ such that
Im g(t_0) = 0. Since Im g(0) = −Im g(t_0) = 0.
Now observe that the vectors x and y=$e^{it_0}y$ are linearly independent. Otherwise $x = \alpha\hat{y}$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and
$0 = (Bx|x) = |\alpha|^2(B\hat{y}—\hat{y})$=$(By|y) = 1$.
Define continuous functions z and f by
$z(b) = \frac{(1-b)x+by}{\|(1-b)x+by\|}$, $b \in [0,1]$ and
$f(b) = (Bz(b)|z(b))$, $b \in [0,1]$. A straight forward calculation shows that f is a real-valued function with $f(0) = 0$ and $f(1) = 1$.
Thus $t \in [0,1] \subset f([0,1]) \subset [0,1] \subset W(B)$ as required.

□

Theorem 4.7. *Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Let $A \in B(H)$ be posinormal then* $\|A\| = \overline{W(A)}$.

Corollary 4.8. *Let $A \in B(H)$ be posinormal then* $0 \in W(A)$.

Proof. Since A is bounded, then every eigenvalue of A that lies on the boundary of $W(A)$ is a normal eigenvalue. An eigenvalue λ is said to be normal for an operator $A \in B(H)$ if

$$Ker(A - \lambda I) = Ker(A^* - \overline{\lambda} I).$$

Let us assume without loss of generality that $\lambda = 0$. Suppose there is a unit vector f for which $Af = 0$ but $A^*f \neq 0$. Let $g = \frac{A^*f}{\|A^*f\|}$. Because $\langle f, A^*f\rangle = \langle Af, f\rangle = \langle 0, f\rangle = 0$ the pair (f,g) is orthonormal in H, and therefore spans a two dimensional subspace M. It follows that $W(A)$ contains the numerical range of the compression A_M of A to M. It is enough to show that 0 is in the interior of $W(A_M)$. Now the matrix of A_M with respect to the orthonormal basis (f,g) of M is of the form $\begin{pmatrix} 0 & a \\ 0 & * \end{pmatrix}$, where $a = \langle A_M g, f\rangle$. We need to show that $a \neq 0$, this will establish $W(A_M)$ as a non degenerate elliptical disk with one focus at 0, and therefore complete the proof. Now,

$$a = \langle A_M g, f\rangle = \langle PAg, f\rangle = \langle Ag, f\rangle = \langle g, A^*f\rangle,$$

where the term on the right, upon recalling that $g = \frac{A^*f}{\|A^*f\|}$, is just

$$\frac{\langle A^*f, A^*f\rangle}{\|A^*f\|} = \|A^*f\| \neq 0,$$

as desired. □

4.3 Spectrum

Lemma 4.9. *Let A be a posinormal operator. If $z \in \sigma_p(A)$ for $0 < p < \frac{1}{2}$, then $\overline{z} \in \sigma_p(A^*)$.*

Proof. Suppose $0 \in \sigma_p(A)$. Then there exists a non-zero vector $x \in H$ such that $Ax = 0$. Since $|A|^2 x = A^*Ax = 0$ and $|A| \geq 0$, we have $(A^*A)^{\frac{1}{2}k} x = 0 (k = 1, 2, ...)$. For $m \in \mathbb{N}$ such that $\frac{1}{m} < p$, we have

$(A^*A)^{\frac{1}{2}m}x = 0$. It follows that $(A^*A)^p x = 0$. Clearly $(AA^*)^p x = 0$ since A is posinormal. Therefore $A^*x = 0$. Next assume that $z \in \sigma_p(A)$ for non-zero $z \in \mathbb{C}$. Then there exists a non-zero vector $y \in H$ such that $Ay = zy$. Let $A = U|A|$ be a polar decomposition of A with unitary operator U. Since $U|A|y = zy$, it follows that $|A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}|A|^{\frac{1}{2}}y = z|A|^{\frac{1}{2}}y$. We know that $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. Hence, we have $\widetilde{A}^* = |A|^{\frac{1}{2}}U^*|A|^{\frac{1}{2}}y = \bar{z}.|A|^{\frac{1}{2}}y$. Thus $A^*(|A|y) = \bar{z}.|A|y$. Since $|A|y \neq 0$, then $\overline{z} \in \sigma_p(A^*)$. □

Theorem 4.10. *Let $A \in B(H)$ be a posinormal operator. Then*

$$\sigma(A) = \{z : \bar{z} \in \sigma_\pi(A^*)\}.$$

Proof. Since we have $\sigma(A) = \sigma_\pi(A) \cup \{z : \bar{z} \in \sigma_\pi(A^*)\}$, it suffices to show that $\sigma(A) = \{z : \bar{z} \in \sigma_\pi(A^*)\}$. Assume that $z \in \sigma_\pi(A)$. Then we have $z \in \pi_p(T(A))$ where T is a mapping. Since $T(A)$ is posinormal we have $\overline{z} \in \sigma_p(T(A^*))$. Also since, $\sigma_p(T(A^*)) = \sigma_\pi(A^*)$, it follows that $\bar{z} \in \sigma_\pi(A^*)$. □

Lemma 4.11. *Let $\mathbb{A} = (A_1, \ldots.., A_n)$ be doubly commuting n-tuple of posinormal operators on H. If $z = (z_1, \ldots.., z_n) \in \sigma_p(\mathbb{A})$, then $\bar{z} = (\bar{z}_1, \ldots.., \bar{z}_n) \in \sigma_p(\mathbb{A}^*)$, where $\mathbb{A}^* = A_1^*, \ldots., A_n^*$.*

Proof. There exists a non-zero vector $x \in H$ such that $A_1x = z_1x(i = 1, \ldots., n)$. We assume that $z_1, \ldots., z_k$ are non-zero and $z_{k+1} = \ldots. = z_n = 0$. therefore we obtain

$$A_{k+1}^* = \ldots = A^*x = 0.$$

Also $A_i^*(A_i|x) = \bar{z}_i.|A_i|x$, where A_{A_i} is the positive operator in a polar decomposition $A_i = U_i|A_i|$ where $i = 1, \ldots., k$. Suppose $|A_1|\ldots.|A_k|x = 0$.

Since $(A_1....A_k)$ is doubly commuting k-tuple of a posinormal operator, then U_i and $|A_i|$ commute with U_j and $|A_j|$ for every $i \neq j$. Thus we have

$$A_1.A_2...A_kx = 0.$$

It follows that $z_1, ..., z_k = 0$. Since every $z_i \neq 0(i = 1, ..., k)$. Therefore we have $|A_1|....|A_k|x \neq 0$. For $i(i = 1, ..., k)$, we have

$$A_i^*(|A_1|...|A_k|x) = |A_1|...|A_{i-1}|.|A_{i+1}|...|A_k|.A_i^*|A_i|x$$

$= \bar{z_i}(|A_i|...|A_k|x)$. Since also A_i commutes with $|A_1...|A_k|$, we have

$$A_i^*(|A_1|...|A_k|x) = 0(i = k + 1, ..., n)$$

Therefore it follows that $\bar{z} = (\bar{z_1}, ..., \bar{z_n}) \in \sigma_p(\mathbb{A}^*)$. □

Theorem 4.12. *Let* $\mathbb{A} = (A_1, ..., A_n)$ *be doubly commuting n-tuple of posinormal operators on* H*. Then*

$$\sigma(\mathbb{A}) = \{(z_1, ..., z_n) \in \mathbb{C}^n : (\bar{z_1}, ..., \bar{z_n}) \in \sigma_\pi(\mathbb{A}^*)\}.$$

Proof. Since $\mathbb{A}$ is a doubly commuting n-tuple it follows that $(z_1, ..., z_n) \in \sigma(\mathbb{A})$. If there exist some partition $\{i_1, ..., i_m\} \cup \{j_1, ..., j_s\} = \{1, ..., n\}$ and a sequence x_k of unit vectors in H such that
$(A_{i_\mu} - z_{i_\mu})x_k \to 0$ and $(A_{j_v} - z_{j_v})^*x_k \to 0$ as $k \to \infty$, for $\mu = 1, ..., m$ and $v = 1, ..., s$. Consider the mapping T such that

$$(z_{i_1}, ..., z_{i_m}, \bar{z}_{j_1}, ..., \bar{z}_{j_s}) \in \sigma_\pi(T(B),$$

where $B = (A_{i_1}, ..., A_{i_m})$

Therefore $T(B) = (T(A_{i_1}, ..., T(A_{i_m}), T(A^*_{j_i}))$. Since $T(A_i)$ is a posinormal operator for every $i = 1, ..., n$ we have $(\bar{z}_1, ..., \bar{z}_n) \in \sigma_p(T(A^*))$. Therefore it follows that $(\bar{z}_1, ..., \bar{z}_n) \in \sigma_\pi(A^*)$. Clearly $\sigma_\pi(A^*) \subset \sigma(A)$ and so

$$\sigma(\mathbb{A}) = \{(z_1, ..., z_n) \in \mathbb{C}^n : (\bar{z_1}, ..., \bar{z_n}) \in \sigma_\pi(\mathbb{A}^*)\}.$$

□

Theorem 4.13. *Let* $\mathbb{A} = (A_1, ..., A_n)$ *be a doubly commuting n-tuple of posinormal operators on* H*. If* $(r_1, ..., r_n) \in \sigma(\mathbb{A}^*\mathbb{A}) \cup \sigma(\mathbb{A}\mathbb{A}^*)$*, then there exists* $(z_1, ..., z_n \in \sigma(\mathbb{A})$ *such that* $|z_i|^2 \geq r_i (i = 1, ..., n)$*, where* $\mathbb{A}^*\mathbb{A} = (A_1^*A_1, ..., A_n^*A_n)$ *and* $\mathbb{A}\mathbb{A}^* = (A_1A_1^*, ..., A_nA_n^*)$.

Proof. We shall prove the theorem by induction. The theorem holds when $n = 1$. We assume that the theorem holds for all doubly commuting $(n-1)-$ tuple of posinormal operators. Assume that $(r_1, ..., r_n) \in \sigma(\mathbb{A}^*\mathbb{A})$. Now since $\sigma(\mathbb{A}^*\mathbb{A}) = \sigma_\pi(\mathbb{A}^*\mathbb{A})$, we have $(\sqrt{r_1}, ..., \sqrt{r_n}) \in \sigma_\pi(|\mathbb{A}|)$, where $|\mathbb{A}| = (|A_1|, ..., |A_n|)$. Consider the mapping $T : B(H) \to B(H)$ such that $\sigma(T(A) = \sigma(A)$ and $\sigma_\pi(A) = \sigma_\pi(T(A)) = \sigma_p(T(A))$ where $\sigma_\pi(A)$ and $\sigma_p(A)$ are the approximate point spectrum and the point spectrum of A, respectively. Let $\mathbb{R} = ker(|T(A_n)| - \sqrt{r_n}) \neq \{0\}$. Then $\mathbb{R}$ is a reducing subspace of $T(A_1), ..., T(A_{n-1})$ and $(T(A_1)_{|\mathbb{R}}, ..., (T(A_{n-1})_{|\mathbb{R}})$ is a doubly commuting $n-1$-tuple of posinormal operators on $\mathbb{R}$. Since $\sum_{i=1}^{n}(|T(A_i)| - \sqrt{r_i})^2$ is not invertible, then

$$ker(\sum_{i=1}^{n}(|T(A_i)| - \sqrt{r_i})^2) = \{\cap_{i=1}^{n-1} ker(|T(A_i)| - \sqrt{r_i})\} \cap \mathbb{R} \neq \{0\}.$$

Hence it follows that $(\sqrt{r_1}, ..., \sqrt{r_{n-1}}) \in \sigma(R)$, where $R = (T(A_i)_{|\mathbb{R}}, ..., T(A_{n-1})_{|\mathbb{R}})$. therefore by the induction hypothesis, there exists $(z_1, ..., z_{n-1}) \in \sigma(S)$ such that $|z_i| \geq \sqrt{r_i} : i = 1, ..., n-1$, where $S = (T(A_i)_{|\mathbb{R}}, ..., T(A_{n-1})_{|\mathbb{R}})$. It thus follows that $(\bar{z}_1, ..., \bar{z}_{n-1}) \in \sigma_p(S^*,)$ there exists a non-zero vector x_0 in $\mathbb{R}$ such that

$$T(A_i^*)x_0 = \bar{z}_i x_0 : i = 1, ..., n-1.$$

Therefore $\sum_{i=1}^{n-1}(T(A_i) - z_i)(T(A_i) - z_i)^* + (|T(A_n)| - \sqrt{r_n})^2$ is not invertible. Hence

$$ker(\sum_{i=1}^{n-1}(T(A_i) - z_i)(T(A_i) - z_i)^* + (|T(A_n)| - \sqrt{r_n})^2) \neq \{0\}.$$

Let $\mathbb{P} = ker(\sum_{i=1}^{n-1}(T(A_i) - z_i)(T(A_i) - z_i)^*$. Then $\mathbb{P}$ reduces $T(A_n)$. Also since $\mathbb{R} \cap \mathbb{P} \neq \{0\}$, $\sqrt{r_n} \in \sigma(|T(A_n)_{|N}|)$. Since $T(A_n)_{|\mathbb{R}}$ is a posinormal operator then there is a $z_n \in \mathbb{C}$ such that $(T(A_n)_{|\mathbb{R}} - z_n)(T(A_n)_{|\mathbb{R}} - z_n)^*$ is not invertible and $|z_n|^2 \geq r_n$.

Since

$$\sum_{i=1}^{n}(T(A_i) - z_i)(T(A_i) - z_i)^*$$

is not invertible, this point $z_1, ..., z_n$ is in $\sigma(\mathbb{A})$ and satisfies

$$|z_i|^2 \geq r_i (i = 1, ..., n).$$

Thus the proof is complete. □

4.4 The relationship between the numerical range and spectrum

In this section we investigate the relationship between the numerical range and the spectrum of a posinormal operator. Our main aim in this section is to show that if A is a posinormal operator then

$$\sigma_p(A) \subseteq \overline{W_p(A)}.$$

Theorem 4.14. *Let $A \in B(H)$ on a complex Hilbert space H be posinormal. Then $\sigma_p(A) \subseteq \overline{W_p(A)}$.*

Proof. If λ is not a member of $\overline{W_p(A)}$, then $d = dist(\lambda, \overline{W_p(A)}) > 0$, (where $dist$ is the distance function derived from the modulus in $\mathbb{C}$) then $\lambda I - A$ has an inverse and $(\|\lambda I - A)^{-1}\| < \frac{1}{d}$. By definition of distance d we have

$$d \leq |\langle Ax, x\rangle - \lambda|, \forall x \in H\|x\| = 1$$

This implies,

$$d\|x\|^2 \leq |(\langle A - \lambda I)x, x\rangle|, \forall x \in H$$

and using the cauchy-Schwatz inequality, we see that

$$(\|A - \lambda I)x\| \geq d\|x\|.$$

Since $(\langle A - \lambda I)$ is bounded below, $(\lambda I - A)^{-1}$ exists on $\mathbf{R}_{(A-\lambda I)}$ and is bounded; moreover

$$\|(A-\lambda I)^{-1}y\| \geq d^{-1}\|y\|, \forall\lambda \in \mathbf{R}_{(A-\lambda I)}$$

Hence, there are only two possibilities, that is $\lambda \in \rho(A)$ or $\lambda \in R\sigma(A)$. Suppose $\lambda \in R\sigma(A)$. Since,

$$\begin{aligned}\overline{R_{(A-\lambda I)}}^{\perp} &= \{R_{A-\lambda I}\}^{\perp} \\ &= ker(A^* - \overline{\lambda} I)(Nullspace).\end{aligned}$$

If $\lambda \in R\sigma(A)$, then $\overline{R_{(A-\lambda I)}}^{\perp} \neq \{0\}$, that is, $ker(A^* - \bar{\lambda} I) \neq \{0\}$.
Hence $\bar{\lambda}$ is an eigenvalue of A^*. If $x \in H$, $\|x\| = 1$ and is such that $A^*x = \bar{\lambda}x$, then $Ax = \lambda x$, for $x \neq 0$

$$\begin{aligned}\langle Ax, x\rangle &= \langle x, A^*x\rangle \\ &= \langle x, \overline{\lambda x}\rangle \\ &= \lambda\langle x, x\rangle \\ &= \lambda\|x\|^2 \\ &= \lambda\end{aligned}$$

This implies that $\lambda \in W_p(A)$, which is a contradiction. Hence, if λ is not a member of $\overline{W_p(A)}$ then λ is not a member of $\sigma_p(A)$, this shows that $\sigma_p(A) \subseteq \overline{W_p(A)}$.
Alternatively, $\lambda \in W_p(A)$ implies that there exists a sequence $\{x_n\}$ of

unit vectors in H such that since for such x_n

$$\begin{aligned} |\lambda - \langle Ax_n, x_n\rangle| &= |\langle(\lambda I - A)x_n, x_n\rangle| \\ &\leq \|(\lambda I - A)x_n\|\|x_n\| \\ &\leq \|(\lambda I - A)x_n\| \to 0 \end{aligned}$$

as $n \to \infty$

Therefore

$$\lambda = lim_{n\to\infty}\langle Ax_n, x_n\rangle.$$

It therefore follows that $\lambda \in W_p(A)$. Since

$$|\lambda| = \|A\| = w(A) = sup\{|\lambda| : \lambda \in \sigma_p(A)\}$$

So $\|A\| \in \sigma_p(A)$ implies that $\|A\| \in \overline{W_p(A)}$, hence

$$\sigma_p(A) \subseteq \overline{W_p(A)}.$$

□

Theorem 4.15. *Let A be posinormal, then $\overline{W_e(A)} = conv(\sigma_e(A))$ if and only if $\forall\lambda \in conv(\sigma_e(A)), \|R_\lambda(A)\| \leq (d(\lambda, conv(\sigma_e(A)))$, where $d = dist(\lambda, \overline{W_e(A)}) > 0$, (dist is the distance function derived from the modulus in $\mathbb{C}$.)*

Proof. We apply the transformation $A \mapsto \alpha A + \beta$ and suppose that

$$[\lambda < 0, 0 \in conv\sigma_e(A) \subset \{z \in \mathbb{C} : Rez \geq 0\}], \forall\lambda < 0.$$

Let $\overline{W_e(A)} = conv(\sigma_e(A))$. Now for all $x \in H$, we have

$$\|(A-\lambda)x\|^2 = \|Ax\|^2 - \lambda[(Ax,x)+(x,Ax)] + \lambda^2\|x\|^2 \geq \lambda^2\|x\|^2$$

Since $(A-\lambda)$ is invertible, we have $\|x\|^2 \geq \lambda^2\|(A-\lambda)^{-1}x\|^2, \forall x \in H$. Hence $|\lambda|^{-1} \geq \|(A-\lambda)^{-1}x\|$, or $|\lambda| = d(\lambda, conv\sigma(A))$.
Conversely, suppose that $\|R_\lambda(A)\| \leq (d(\lambda, Conv\sigma_e(A)))$. We need to prove that $\overline{W_e(A)} = conv(\sigma_e(A))$. It suffices to show that if λ is not in the convex hull of $\sigma_e(A)$, then also λ is not in $\overline{W_e(A)}$.
By applying the transformation $A \mapsto \alpha A + \beta$ we can assume that

$$[\lambda < 0, 0 \in conv\sigma_e(A) \subset \{z \in \mathbb{C} : Rez \geq 0\}], \forall \lambda < 0.$$

The estimate $dist(c, Conv\sigma_e(A)) \geq |c|$ implies $\|(A-c)^{-1}\| \leq |c|^{-1}$, so

$$c^2\|x\|^2 \leq ((A-c)x|(A-c)x).$$

Let c tend to infinity, therefore

$$(Ax|x) + (x|Ax) \geq 0.$$

Hence,

$$\overline{W_e(A)} \subset \{z \in \mathbb{C} : Rez \geq 0\},$$

that is, λ is not in $\overline{W_e(A)}$ as desired. □

Theorem 4.16. *Let A be a posinormal operator on H. Then $\sigma_e(A) \subseteq W_e(A)$.*

Proof. Let $\sigma_e(A) \in W_e(A)$ and let $B = \lambda I_H - A \forall A \in B(H)$. There are three cases: the range of B is not closed, the kernel of B is infinite dimensional, or the kernel of B^* is infinite dimensional.

If the range of B is not closed, then B is not bounded below on the orthogonal complement of $ker(B)$. Let $X = ker(B)^{\perp}$. Then there exists a unit vector $x_1 \in X$ such that $\|Bx_1\| \leq 1$. Then, since B is not bounded below, there must exist a unit vector $x_2 \in X$ orthogonal to x_1 such that $\|Bx_2\| \leq \frac{1}{2}$. Repeating this process gives us an orthonormal sequence $\{x_n\}_{n\geq 1}$ such that $lim_{n\to\infty}\|Bx_n\| = 0$. Thus $\lambda \in \sigma_e(A)$.

If the kernel of B is infinite dimensional, we can easily construct an orthonormal sequence $\{x_n\}_{n\geq 1}$ such that $\langle Bx_n, x_n\rangle = 0$ for all n. In the same way if the kernel of B^* is infinite dimensional then $\bar{\lambda} \in W_e(A^*)$. We know that $W_e(A^*) = \overline{W_e A}$ therefore $\bar{\lambda} \in \overline{W_e A}$ hence $\lambda \in W_e(A)$ and thus $\sigma_e(A) \subseteq W_e(A)$.

□

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this last chapter, conclusions are drawn and recommendations made based on the objectives of the study and the results obtained.

5.2 Conclusion

Results for characterization of numerical ranges and spectra of posinormal operators have been obtained in this study.The aim of this study has been to investigate the numerical ranges and spectra of posinormal operators and to establish the relationship of the numerical range and spectrum of a posinormal operator.We investigated the numerical range of a bounded posinormal operator A acting on a complex Hilbert space H and showed that it is an ellipse whose foci are the eigenvalues of A and it possesses the following properties: the numerical range of A is nonempty; $W(A)$ is always convex;Zero is contained in the numerical range of A, i.e $0 \in W(A)$; the norm of A is a subset of the closure of $W(A)$ and the numerical radius of A is equal to the norm of A.

Then we investigated the spectrum of a bounded posinormal operator A acting on a complex Hilbert space H and proved that it satisfies Xia's property, i.e $\sigma(A) = \{z : \bar{z} \in \sigma_\pi(A^*)\}$. For a posinormal operator A if z is a member of the point spectrum of A the the closure of z is a member of the point spectrum of the adjoint of A. Lastly, doubly commuting n-tuples of posinormal operators are jointly normaloid.

Finally the study has established the relationship between the numerical range and spectrum of a posinormal operator. The following results describes the relationship between the numerical range and spectrum of a posinormal operator: For a posinormal operator $A \in B(H)$ on a complex Hilbert space H, $\sigma_p(A) \subseteq \overline{W_p(A)}$; $\overline{W_e(A)} = conv(\sigma_e(A))$; and $\sigma_e(A) \subseteq W_e(A)$.

Therefore, the main results of the numerical ranges and spectra of posinormal operators and the relationship between the numerical range and spectrum of a posinormal operator obtained in this study are in line with the stated objectives.

5.3 Recommendations

The results obtained are specific to the numerical ranges and spectra of posinormal operators on a complex Hilbert space. It is interesting to investigate and characterize the numerical ranges and spectra of polynomially posinormal operators and coposinormal operators. It is also important to establish the relationship between the numerical range and spectrum of these Hilbert space operators.

Further research can be done on characterization of the numerical ranges and spectra of posinormal operators when the Hilbert space is dense and non-separable. Other properties of posinormal operators, for example norms, can also be characterized.

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